Consider the linear system of first order differential equations $\mathbf{x}' = A\mathbf{x}$, where $\mathbf{x} = \mathbf{x}(t)$, $t \geq 0$, and A has the eigenvalues and eigenvectors below.

$$\lambda_1 = -2,\ \mathbf{v}_1 = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}, \quad \lambda_2 = -3,\ \mathbf{v}_2 = \begin{pmatrix} 4 \\ 0 \\ 7 \end{pmatrix}, \quad \lambda_3 = -3,\ \mathbf{v}_3 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

i) Identify three solutions to the system, $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, and $\mathbf{x}_3(t)$.

ii) Use a determinant to identify values of t, if any, where $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ form a fundamental set of solutions. You may assume that they are solutions to $\mathbf{x}' = A\mathbf{x}$.

iii) Given the initial value $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, give a solution to the IVP. Please show your work.

Question: Consider the linear homogeneous system x' = p11(t)x + p12(t)y, y' = p21(t)x + p22(t)y. Show that if x = x1(t), y = y1(t) and x = x2(t), y = y2(t) are two...

Consider the linear homogeneous system

$x' = p_{11}(t)x + p_{12}(t)y,$

$y' = p_{21}(t)x + p_{22}(t)y.$

Show that if $x = x_1(t)$, $y = y_1(t)$ and $x = x_2(t)$, $y = y_2(t)$ are two solutions of the given system, then $x = c_1x_1(t) + c_2x_2(t)$, $y = c_1y_1(t) + c_2y_2(t)$ is also a solution for any constants c_1 and c_2. This is the principle of superposition.

Ex 3 (#3, p.389) Show that the Wronskians of two fundamental sets of solutions of the system

$$(3) \qquad \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}' = \begin{bmatrix} p_{11}(t) & \cdots & p_{1n}(t) \\ \vdots & & \vdots \\ p_{n1}(t) & \cdots & p_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

on an interval $\alpha < t < \beta$ can differ at most by a multiplicative constant.

Let $\tilde{x}^{(1)}(t), \ldots, \tilde{x}^{(n)}(t)$ and $\tilde{z}^{(1)}(t), \ldots \tilde{z}^{(n)}(t)$ be two fundamental set of sol. to (3) on $\alpha < t < \beta$, and let.

$$W_1(t) = \det[\tilde{x}^{(1)}t \ \ldots \ \tilde{x}^{(n)}t]$$

$$W_2(t) = \det[\tilde{z}^{(1)}t \ \ldots \ \tilde{z}^{(n)}t]$$

We also know that there exist constants C_1 and C_2 such that

$$W_1(t) = C_1 e\left\{ \int_{t_0}^{t} [p_{11}x + p_{22}x + \ldots p_{nn}x] dx \right\}$$

and

$$W_2(t) = C_2 e\left\{ \int_{t_0}^{t} [p_{11}x + p_{22}x + \ldots p_{nn}x] dx \right\}$$

for all $\alpha < t < \beta$. The formula W_1 and W_2 is only different at most by a multiplicative constant on $\alpha < t < \beta$.

2. Consider the following system of differential equations:

$$x'' + x - 2y = t^2$$
$$y'' - x = e^t$$

Using the method of elimination, obtain a single differential equation for x (in other words with no y terms involved). *Do not solve the resulting equation.*

account for second derivative → $x'' = D^2x$, $y'' = D^2y$

$$\begin{aligned} x'' + x - 2y &= t^2 \\ -x + y'' &= e^t \end{aligned} \quad \begin{aligned} (D^2+1)x - 2y &= t^2 \\ (-1)x + D^2y &= e^t \end{aligned}$$

$$\begin{pmatrix} D^2+1 & -2 \\ -1 & D^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t^2 \\ e^t \end{pmatrix}$$

3. Consider the following system: $t^2 x' = \begin{pmatrix} 0 & 2t^2 \\ 1 & 1 \end{pmatrix} x$.

(b) Find a, b, c, d so that $x_2 = \begin{pmatrix} at^2 + bt + 1 \\ ct + d \end{pmatrix}$ is a solution to the system.

General form of $\frac{dx}{dt} = \begin{pmatrix} 10 & -5 \\ 5 & 0 \end{pmatrix} x$

Eigen Value → $\det(A - \lambda I) = 0$

$$\lambda^2 - 10\lambda + 25 = (\lambda - 5)^2$$

$\lambda = 5$

repeated

find Eigen Vector ?

$$(A - \lambda I)\vec{v} = 0$$

$$\begin{vmatrix} 10-5 & -5 \\ 5 & 0-5 \end{vmatrix} \begin{pmatrix} \vec{v}_1 \\ \vec{v}_2 \end{pmatrix} = \vec{0}$$

$$5\vec{v}_1 - 5\vec{v}_2 = 0$$
$$5\vec{v}_1 - 5\vec{v}_2 = 0$$
$$\vec{v}_1 = \vec{v}_2$$

let $\vec{v}_1 = 1$ then $\vec{v}_2 = 1$

Hence $x_{(1)}t = C_1 e^{5t}\begin{pmatrix}1\\1\end{pmatrix}$

for repeated root

find w such that $(A-\lambda I)\vec{w} = \vec{v}$

$$\begin{vmatrix} 5 & -5 \\ 5 & -5 \end{vmatrix} \begin{matrix}\tilde{w}_1\\ \tilde{w}_2\end{matrix} = \begin{pmatrix}1\\1\end{pmatrix}$$

$$5\tilde{w}_1 - 5\tilde{w}_2 = 1$$

$$-\ 5\tilde{w}_1 - 5\tilde{w}_2 = 1$$

$$5\tilde{w}_1 = 5\tilde{w}_2 + 1$$

let $\tilde{w}_1 = 1$

then

$$5-1 = 5\tilde{w}_2$$

$$\tilde{w}_2 = \frac{4}{5}$$

4. Find the general form for solutions of the system $\frac{dx}{dt} = \begin{pmatrix} 10 & -5 \\ 5 & 0 \end{pmatrix} x$.

$$X' = AX$$

$A = \begin{pmatrix} 10 & -5 \\ 5 & 0 \end{pmatrix} \Rightarrow$ find Eigen Value $\Rightarrow \lambda^2 - 10 + 25 = 0$

$$\downarrow$$

$$(\lambda - 5)^2 = 0$$

$\lambda = 5 \rightarrow$ repeated root

Case $\lambda = 5$ $\quad (A - \lambda I)\tilde{v} = 0$

Row reduce $\sim \begin{pmatrix} 5 & -5 \\ 5 & -5 \end{pmatrix} \begin{pmatrix} \tilde{v} \end{pmatrix} = 0$

$$5\tilde{v}_1 - 5\tilde{v}_2 = 0$$

$$5\tilde{v}_1 = 5\tilde{v}_2$$

let $\tilde{v}_1 = \tilde{v}_2 = 1$

$$X^{(1)} = e^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

fore repeated value $\quad X^{(2)} t = e^{\lambda t}(t\tilde{v}_1 + W)$

where $\quad (A - \lambda I)W = \tilde{v}_1$

$$\downarrow$$

$$\begin{pmatrix} 5 & -5 \\ 5 & -5 \end{pmatrix} \tilde{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$5\tilde{w}_1 - 5\tilde{w}_2 = 1$$

$$5\tilde{w}_1 - 5\tilde{w}_2 = 1$$

$$5\tilde{w}_1 = 1 + 5\tilde{w}_2$$

let $\tilde{w}_2 = 1$

then $\tilde{w}_1 = \frac{6}{5}$

$$x^{(2)}t = t\begin{pmatrix}1\\1\end{pmatrix}e^{5t} + \begin{pmatrix}\frac{6}{5}\\1\end{pmatrix}e^{5t}$$

$$\boxed{\text{General Form} \Rightarrow x(t) = C_1\begin{pmatrix}1\\1\end{pmatrix}e^{5t} + C_2\left(t\begin{pmatrix}1\\1\end{pmatrix}e^{5t} + \begin{pmatrix}\frac{6}{5}\\1\end{pmatrix}e^{5t}\right)}$$

$$C_2 e^{5t}\begin{pmatrix}t+\frac{6}{5}\\t\end{pmatrix}$$

5. Solve for x(t) and y(t) given

$$x' = 2x - 9y$$
$$y' = x + 2y$$

with initial conditions x(0) = 0 and y(0) = 1.

$$z' = \begin{pmatrix}x'\\y'\end{pmatrix}$$

Initial

as $= z = \begin{pmatrix}x\\y\end{pmatrix}$

$$z(0) = \begin{pmatrix}0\\1\end{pmatrix}$$

$$z' = Az$$

$$x' = \begin{pmatrix}2 & -9\\1 & 2\end{pmatrix}x$$

Eigen Value $\Rightarrow$ $\lambda^2 - 4\lambda + 13 = 0$

$$\frac{4 \pm \sqrt{16 - 4(1)(13)}}{2}$$

$$= \frac{4 \pm \sqrt{16 - 52}}{2} \Rightarrow \frac{4 \pm \sqrt{-36}}{2}$$

$$\begin{array}{r}13\\ \times 4\\ \hline 52\end{array}$$

$$\left.\begin{aligned} x' &= 2x - 9y \\ y' &= x + 2y \end{aligned}\right\} \quad \begin{vmatrix} 2 & -9 \\ 1 & 2 \end{vmatrix}$$

$$\downarrow$$

Eigen Value $\Rightarrow$ $\det(A - \lambda I) = 0$

$$= \lambda^2 - 4\lambda + 13 = 0$$

$$\begin{array}{r} 13 \\ \times\ 4 \\ \hline 52 \end{array}$$

$$\downarrow$$

$$\frac{4 \pm \sqrt{16 - 4(1)(13)}}{2}$$

$$= \frac{4 \pm \sqrt{16 - 52}}{2} \rightarrow$$

$$\begin{array}{r} 52 \\ -\ 16 \\ \hline 36 \end{array}$$

$$\frac{4 \pm \sqrt{-36}}{2} = 2 \pm 3i$$

$$\lambda = 2 \pm 3i$$

$$\downarrow$$

let use case $\lambda = 2 + 3i$

$$\begin{vmatrix} 2 - (2+3i) & -9 \\ 1 & 2 - (2+3i) \end{vmatrix} \left(\vec{v} \right) = \vec{0}$$

$$\begin{vmatrix} -3i & -9 \\ 1 & -3i \end{vmatrix} \begin{pmatrix} \vec{v}_1 \\ \vec{v}_2 \end{pmatrix} = \vec{0}$$

$$-3i\vec{v}_1 - 9\vec{v}_2 = 0$$

$$\vec{v}_1 - 3i\vec{v}_2 = 0 \rightarrow \vec{v}_1 = 3i\vec{v}_2$$

$$\downarrow$$

let $\vec{v}_2 = 1$ then $\vec{v}_1 = 3i$

Hence $$y = e^{2t}(\cos(3t) + i\sin(3t))\begin{pmatrix} 3i \\ 1 \end{pmatrix}$$

$$= e^{2t}\begin{pmatrix} 3i\cos(3t) - 3\sin(3t) \\ \cos(3t) + i\sin(3t) \end{pmatrix}$$

$$= e^{2t}\begin{pmatrix} -3\sin(3t) \\ \cos(3t) \end{pmatrix} + ie^{2t}\begin{pmatrix} 3\cos(3t) \\ \sin(3t) \end{pmatrix}$$

$$y = c_1 e^{2t}\begin{pmatrix} -3\sin(3t) \\ \cos(3t) \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 3\cos(3t) \\ \sin(3t) \end{pmatrix}$$

$$\frac{4 \pm 6i}{2}$$

$$= 2 \pm 3i$$

use case $\lambda = 2 + 3i$

Find Eigen Vector $\Rightarrow$ $(A - \lambda I)\vec{v} = 0$

$$\begin{pmatrix} 2-2-3i & -9 \\ 1 & 2-2-3i \end{pmatrix} \vec{v} = 0$$

$$\begin{pmatrix} -3i & -9 \\ 1 & -3i \end{pmatrix} \vec{v} = 0$$

$$-3i\vec{v}_1 - 9\vec{v}_2 = 0$$

$$1\vec{v}_1 - 3i\vec{v}_2 = 0$$

let $\vec{v}_1 = 3i$, $\vec{v}_2 = 1$

$$v = \begin{pmatrix} 3i \\ 1 \end{pmatrix}$$

$$x^{\lambda t} = e^{2t}(\cos(3t) + i\sin(3t))\begin{pmatrix} 3i \\ 1 \end{pmatrix}$$

$$= e^{2t}\begin{pmatrix} 3i\cos(3t) & -3\sin(3t) \\ \cos(3t) & + i\sin(3t) \end{pmatrix}$$

$$= e^{2t}\begin{pmatrix} -3\sin(3t) \\ \cos(3t) \end{pmatrix} + ie^{2t}\begin{pmatrix} 3\cos(3t) \\ \sin(3t) \end{pmatrix}$$

$$\boxed{x(t) = c_1 e^{2t}\begin{pmatrix} -3\sin(3t) \\ \cos(3t) \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 3\cos(3t) \\ \sin(3t) \end{pmatrix}}$$

As $\quad Z(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\downarrow$

Hence $x(0) = c_1\begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{matrix} 0 \\ 1 \end{matrix}$

$c_2 = 0$

$c_1 = 1$

Sol $\Rightarrow \quad \boxed{x(t) = e^{2t}\begin{pmatrix} -3\sin(3t) \\ \cos(3t) \end{pmatrix}}$ $\begin{matrix} \to x \\ \to y \end{matrix}$

$\Downarrow$

$x = -3e^{2t}\sin(3t)$

$y = e^{2t}\cos(3t)$

6. Compute e^{At} where $A = \begin{pmatrix} 6 & -7 \\ 1 & -2 \end{pmatrix}$.

$-12+7$

Find Eigen Value => $\lambda^2 - 4\lambda - 5 = 0$

$$\downarrow$$

$$(\lambda-5)(\lambda+1)$$

$$\lambda = 5, -1$$

Eigen Vector => $(A - \lambda I)\vec{v} = 0$

Case $\lambda = 5$

$$\begin{pmatrix} 1 & -7 \\ 1 & -7 \end{pmatrix} \vec{v}_1 = 0$$

$$\vec{v}_1 - 7\vec{v}_2 = 0$$

$$\vec{v}_1 = 7\vec{v}_2$$

$$\downarrow$$

let $\vec{v}_2 = 1$, $\vec{v}_1 \Rightarrow 7$

$$\boxed{x^{(1)}t = e^{5t}\begin{pmatrix} 7 \\ 1 \end{pmatrix}}$$

Case $\lambda = -1$

$$\begin{pmatrix} 7 & -7 \\ 1 & -1 \end{pmatrix} \vec{v} = 0$$

$$7\vec{v}_1 - 7\vec{v}_2 = 0 \rightarrow 7\vec{v}_1 = 7\vec{v}_2$$

$$\downarrow$$

let $\vec{v}_1 = \vec{v}_2 = 1$

$$\boxed{x^{(2)}t = e^{-t}\begin{pmatrix}1\\1\end{pmatrix}}$$

4. If $x_1 = y$ and $x_2 = y'$, then the second order equation

$$y'' + p(t)y' + q(t)y = 0 \qquad \text{(i)}$$

corresponds to the system

$$x_1' = x_2,$$
$$x_2' = -q(t)x_1 - p(t)x_2. \qquad \text{(ii)}$$

Show that if $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are a fundamental set of solutions of Eqs. (ii), and if $y^{(1)}$ and $y^{(2)}$ are a fundamental set of solutions of Eq. (i), then $W[y^{(1)}, y^{(2)}] = cW[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}]$, where c is a nonzero constant.
Hint: $y^{(1)}(t)$ and $y^{(2)}(t)$ must be linear combinations of $x_{11}(t)$ and $x_{12}(t)$.

5. Show that the general solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$ is the sum of any particular solution $\mathbf{x}^{(p)}$

Since both of the pair are solution to the same differential equation. Hence $y^{(1)}$ and $y^{(2)}$ must be a linear combination of the solution x_1.

let $y^{(1)} = a_{11}x_{11} + a_{12}x_{12}$

$y^{(2)} = a_{21}x_{11} + a_{22}x_{12}$

where a is also a constant $\Rightarrow$ $W(y^{(1)}, y^{(2)}) \Rightarrow \begin{vmatrix} y^{(1)}, & y^{(2)} \\ y'^{(1)}, & y'^{(2)} \end{vmatrix}$

$$W = \begin{vmatrix} a_{11}x_{11} + a_{12}x_{12} & a_{21}x_{11} + a_{22}x_{12} \\ a_{11}x'_{11} + a_{12}x'_{12} & a_{21}x'_{11} + a_{22}x'_{12} \end{vmatrix}$$

$W = (a_{11}a_{12} - a_{12}a_{21})x_{11}x'_{12} - (a_{11}a_{22} - a_{12}a_{21})x_{12}x'_{11}$

$x_2 = y' = x'_1$

Hence

$$W = (a_{11}a_{22} - a_{12}a_{21})\,x_{11}x_{22} - (a_{11}a_{22} - a_{12}a_{21})\,x_{12}x_{21}$$

$$= (a_{11}a_{22} - a_{12}a_{21})(x_{11}x_{22} - x_{12}x_{21})$$

$$\Rightarrow \underbrace{(a_{11}a_{22} - a_{12}a_{21})}_{c}\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}$$

Prove $\rightarrow$ $\boxed{W[y^{(1)}, y^{(2)}] = cW[x^{(1)}, x^{(2)}]}$

8. Let $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)}$ be solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on the interval $\alpha < t < \beta$. Assume that $\mathbf{P}$ is continuous, and let t_0 be an arbitrary point in the given interval. Show that $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)}$ are linearly dependent for $\alpha < t < \beta$ if (and only if) $\mathbf{x}^{(1)}(t_0), \dots, \mathbf{x}^{(m)}(t_0)$ are linearly dependent. In other words $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)}$ are linearly dependent on the interval (α, β) if they are linearly dependent at any point in it.

Hint: There are constants $c_1, \dots, c_m$ that satisfy $c_1\mathbf{x}^{(1)}(t_0) + \cdots + c_m\mathbf{x}^{(m)}(t_0) = \mathbf{0}$. Let $\mathbf{z}(t) = c_1\mathbf{x}^{(1)}(t) + \cdots + c_m\mathbf{x}^{(m)}(t)$, and use the uniqueness theorem to show that $\mathbf{z}(t) = \mathbf{0}$ for each t in $\alpha < t < \beta$.

as $x^{(1)}(t_0), \dots x^{(m)}(t_0)$ are linreary indlependent at t_0. This means there exist a constant $c_1, \dots, c_m$ not all zero suchthat

$$c_1 x^{(1)}(t_0) + c_2 x^{(2)}(t_0) + \cdots c_m x^{(m)}(t_0) = 0$$

Define => $z(t) = c_1 x^{(1)}(t) + c_2 x^{(2)} t + c_m x^{(m)}(t_0)$ → Hence also $z(t_0)$ sholud be $= 0$

or $x^{(i)'}(t) = P(t) x^{(i)} t$

$$z'(t) = c_1 x'^{(1)} t + c_2 x'^{(2)} t + \cdots c_m x'^{(m)}(t)$$

As $x' = P(t)\, x(t)$

$$z'(t) = c_1 (P(t))\, x^{(1)}(t) + c_2\, P(t)\, x^{(2)}(t)$$

$$= z'(t) = P(t)\, (c_1 x^{(1)}(t)) + c_2 x^{(2)}(t))$$

Sastify as $\boxed{z'(t) = P(t)\, z(t)}$ $\Leftrightarrow x' = P(t)\, x$

9. Let $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ be linearly independent solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, where $\mathbf{P}$ is continuous on $\alpha < t < \beta$.

(a) Show that any solution $\mathbf{x} = \mathbf{z}(t)$ can be written in the form

$$\mathbf{z}(t) = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t)$$

for suitable constants $c_1, \ldots, c_n$.
Hint: Use the result of Problem 12 of Section 7.3, and also Problem 8 above.

(b) Show that the expression for the solution $\mathbf{z}(t)$ in part (a) is unique; that is, if $\mathbf{z}(t) = k_1\mathbf{x}^{(1)}(t) + \cdots + k_n\mathbf{x}^{(n)}(t)$, then $k_1 = c_1, \ldots, k_n = c_n$.
Hint: Show that $(k_1 - c_1)\mathbf{x}^{(1)}(t) + \cdots + (k_n - c_n)\mathbf{x}^{(n)}(t) = \mathbf{0}$ for each t in $\alpha < t < \beta$, and use the linear independence of $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$.

(a.) As $z(t) = C_1 x^{(1)} t + \ldots + C_n x^{(n)} t$

then $\to z'(t) = C_1 x'^{(1)} t + \ldots C_n x'^{(n)} t$

we know that

$\to x'(t) = P(t) x$ so $\to z'(t) = C_1 P(t) x^1 + C_n P(t) x^{(n)}$

$z'(t) = P(t)(C_1 x^1 + C_n x^n)$

$= z'(t) = P(t) z(t)$

Prove.

(b.) $\to$ We know that $\to z(t) = C_1 x^{(1)}(t) + \ldots C_n x^{(n)}(t)$

also $z(t) = K_1 x^{(1)} t + \ldots K_n x^{(n)} t$ Apply initial Condition $z(t_0) = 0$

As both of eq are solution hence $\Rightarrow$ $C_1 x^{(1)} t + \ldots C_n x^{(n)} t = K_1 x^{(1)} t + \ldots K_n x^{(n)} t$

then $(C_1 - K_1) x^{(1)} t + \ldots (C_n - K_n) x^{(n)} t = 0$

Since it is linearly indenpendent then $\boxed{C_i - K_i = 0} \to$ for all $i = 1, 2, \ldots$

Consider the vectors $\mathbf{x}^{(1)}(t) = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$ and $\mathbf{x}^{(2)}(t) = \begin{pmatrix} e^t \\ e^t \end{pmatrix}$, and answer the same questions as in Problem 6.

(a) Compute the Wronskian of $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$.

(b) In what intervals are $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ linearly independent?

(c) What conclusion can be drawn about the coefficients in the system of homogeneous differential equations satisfied by $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$?

(d) Find this system of equations and verify the conclusions of part (c).

(a.) $W[x^{(1)}, x^{(2)}] = \begin{pmatrix} t^2 & e^t \\ 2t & e^t \end{pmatrix} = t^2e^t - 2te^t \neq 0 \rightarrow (t^2-2t)e^t$

(b.) Linear Independant througth out the whole interval except when $t=0$. and $t=2$

(c.) It should have some discontinuity when $t=0$

(d.) $\Rightarrow$ $x' = Ax \Rightarrow \begin{pmatrix} 2t \\ 2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \underbrace{\begin{pmatrix} t^2 \\ 2t \end{pmatrix}}_{X_1}$; $\begin{pmatrix} e^t \\ e^t \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \underbrace{\begin{pmatrix} e^t \\ e^t \end{pmatrix}}_{X_2}$

$2t = at^2 + b2t \rightarrow$ eq1

$2 = ct^2 + d2t \rightarrow$ eq3.

$ae^t + be^t = e^t \rightarrow$ eq2

$ce^t + de^t = e^t \rightarrow$ eq4.

$at^2 = 2t - b(2t)$

$a = \frac{2}{t} - \frac{2b}{t}$

Plug to eq.2 $\rightarrow$

$\left(\frac{2}{t} - \frac{2b}{t}\right)e^t + be^t = e^t$

$\frac{2}{t}e^t - \frac{2b}{t}e^t + be^t = e^t$

$-\frac{2b}{t}e^t + be^t = e^t - \frac{2}{t}e^t$

$b\left(\cancel{-\frac{2}{t}e^t + e^t}\right) = \left(\cancel{e^t - \frac{2}{t}e^t}\right)$

$b = 1$

as $b=1$ then $at^2 = 2t - 2t$

$a = 0$

eq.3:

$2 = ct^2 + d2t \longrightarrow$

$ct^2 = 2 - 2t\,d$

$c = \frac{2}{t^2} - \frac{2d}{t}$

Sub to eq4: $\longrightarrow$ $\left(\frac{2}{t^2} - \frac{2d}{t}\right)\cancel{e^t} + d\cancel{e^t} = \cancel{e^t}$

$2 - 2dt + dt^2 = t^2$

$2 - t^2 = 2dt - dt^2$

$2 - t^2 = d(2t - t^2)$

$d = \frac{2 - t^2}{2t - t^2}$ $\rightarrow$ or $\frac{t^2 - 2}{t^2 - 2t}$

Hence $c \Rightarrow$ $2 = ct^2 + \left(\frac{2 - t^2}{2t - t^2}\right)2t$

Solve give $c = \frac{2 - 2t}{t^2 - 2t}$

x'

2. $\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \mathbf{x}$

$-4-(-6)$
$-4+6=2$

$x' = Ax \longrightarrow$ Eigen Value $\det(A-\lambda I)\vec{v}=0$

or use $\lambda^2+3\lambda+2=0$

$\downarrow$

$(\lambda+2)(\lambda+1)$

Case $\lambda=-2$

$(A-\lambda I)\vec{v}=0$

$\begin{pmatrix} 3 & -2 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} \vec{v}_1 \end{pmatrix}=0$

$3\vec{v}_1-2\vec{v}_2=0$

$3\vec{v}_1=2\vec{v}_2$

let $\vec{v}_1=2 \quad \vec{v}_2=3$

$x^{(1)}t = e^{-2t}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Case $\lambda=-1$

$(A-\lambda I)\vec{v}=0$

$\begin{pmatrix} 2 & -2 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} \vec{v}_1 \end{pmatrix}=0$

$2\vec{v}_1-2\vec{v}_2=0$

$2\vec{v}_1=2\vec{v}_2$

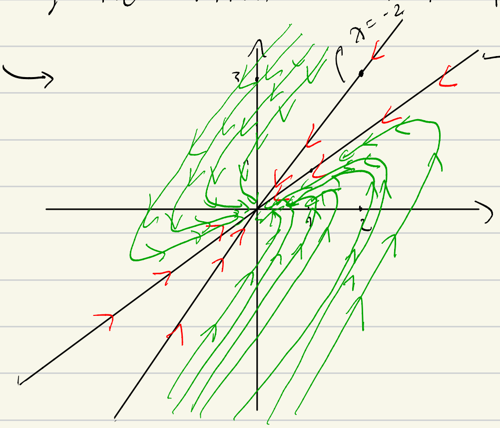
let $\vec{v}_1=\vec{v}_2=1$

$x^{(2)}t = e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

General Sol $\longrightarrow$ $\boxed{x(t) = c_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}}$

as $t \to \infty$ it is nodal sink and Stable
as in the long run ; the solution $x(t)$ will approach $(0,0)$ or (x_0, y_0)

Draw direction field. $\longrightarrow$



6. $\mathbf{x}' = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} \end{pmatrix} \mathbf{x}$

$\frac{25}{4} - \frac{16}{4} = \frac{9}{4}$

$\lambda^2 - \frac{5}{2}\lambda + 1$

$= \dfrac{\frac{5}{2} \pm \sqrt{(\frac{-5}{2})^2 - 4(1)(1)}}{2} = \dfrac{\frac{5}{2} \pm \sqrt{\frac{25}{4} - 4}}{2}$

$= \dfrac{\frac{5}{2} \pm \sqrt{\frac{9}{4}}}{2} = \dfrac{\frac{5}{2} \pm \frac{3}{2}}{2}$

$\lambda = \frac{5}{4} \pm \frac{3}{4}$

$\lambda = 2$ or $\lambda = \frac{1}{2}$

Eigen Value

Find Eigen Vector. $\Rightarrow (A-\lambda I)\vec{v}=0$ $\quad \frac{5}{4}-2 \quad \frac{5}{4}-\frac{1}{2}=\frac{3}{4}$

$$\begin{pmatrix} -\frac{3}{4} & \frac{3}{4} \\ \frac{3}{4} & -\frac{3}{4} \end{pmatrix}\vec{v}=0$$

$$-\frac{3}{4}\vec{v}_1+\frac{3}{4}\vec{v}_2=0$$

$$\downarrow$$

$$\vec{v}_1=\vec{v}_2=1$$

$$x^{(1)}t = e^{2t}\begin{pmatrix}1\\1\end{pmatrix}$$

Case $\lambda=\frac{1}{2}$

$$\begin{pmatrix} \frac{3}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{3}{4} \end{pmatrix}\vec{v}=0$$

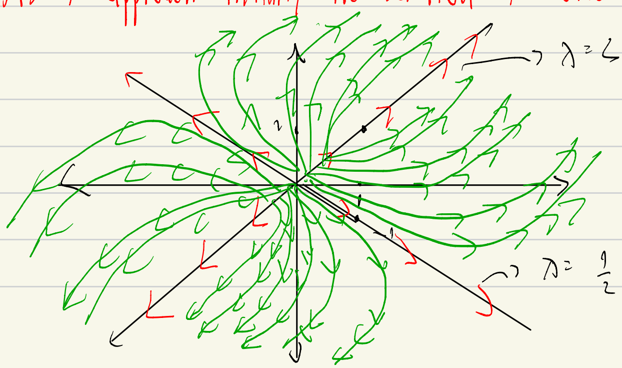
$$\frac{3}{4}\vec{v}_1+\frac{3}{4}\vec{v}_2=0$$

let $\vec{v}_1=1, \vec{v}_2=-1$

$$x^{(2)}t = e^{\frac{1}{2}t}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\boxed{x(t) = c_1\begin{pmatrix}1\\1\end{pmatrix}e^{2t} + c_2\begin{pmatrix}1\\-1\end{pmatrix}e^{\frac{1}{2}t}}$$

As t approach infinity the solution is unbound and $\rightarrow\infty$



Nodal source (Unstable)

11. $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \mathbf{x}$

$x' = Ax \Rightarrow$ find Eigen Value $\det(A - \lambda I) = 0$

$$\begin{pmatrix} 1-\lambda & 1 & 2 \\ 1 & 2-\lambda & 1 \\ 2 & 1 & 1-\lambda \end{pmatrix}$$

$$= (1-\lambda)((2-\lambda)(1-\lambda) - 1) - 1((1-\lambda)-2) + 2(1-2(2-\lambda))$$

$$= (1-\lambda)((2-2\lambda-\lambda+\lambda^2)-1) - 1(-1-\lambda) + 2(-3+2\lambda)$$

$$= (1-\lambda)(\lambda^2-3\lambda+1) + \lambda + 1 - 6 + 4\lambda$$

$$\Rightarrow \cancel{\lambda^2} \cancel{-3\lambda} + 1 \cancel{-\lambda^3} \cancel{+3\lambda^2} \cancel{-\lambda} + \cancel{\lambda} + 1 - 6 + \cancel{4\lambda} = 0$$

$$-\lambda^3 + 4\lambda^2 + \lambda - 4 = 0$$

$$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$$

let $\lambda = 1$ $\quad 1 - 4 - 1 + 4$

$-4 + 4 = 0$

one root is $\lambda = 1$

$$\begin{array}{r|rrrr} 1 & 1 & -4 & -1 & 4 \\ & & 1 & -3 & -4 \\ \hline & 1 & -3 & -4 & 0 \end{array}$$

$$\lambda^2 - 3\lambda - 4$$

$$(\lambda - 4)(\lambda + 1)$$

Eigen Value => $= 1, 4, -1$

<u>Case $\lambda = 1$</u> => $(A - \lambda I)\vec{v} = 0$

$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} \vec{v} \end{pmatrix} = 0$$

$$v_2 + 2v_3 = 0$$
$$v_1 + v_2 + v_3 = 0$$
$$2v_1 + v_2 = 0$$

$$v_2 = -2v_3$$

set $v_3 = 1 \to v_2 = -2$

$2v_1 - 2 = 0 \to v_1 = 1$

$$x^{(1)}t = e^{t}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

<u>Case $\lambda = 4$.</u>

$$\begin{pmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{pmatrix} \vec{v}_1 = 0$$

$$2R_2 - R_3 \quad \begin{pmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 0 & -5 & 5 \end{pmatrix} \vec{v} = 0$$

$$-3v_1 + v_2 + 2v_3 = 0$$
$$v_1 - 2v_2 + v_3 = 0$$

$$-5\check{v}_2+5\check{v}_3=0$$
$$\check{v}_2=1 \quad \check{v}_3=1$$

$$-3\check{v}_1+1+2=0$$
$$-3\check{v}_1+3=0$$
$$\check{v}_1=1$$

$$x^{(2)}t= e^{4t}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

Case $\lambda=-1$

$$\begin{pmatrix}2&1&2\\1&3&1\\2&1&2\end{pmatrix}\check{v}=0$$

$$2R_2-R_3 \sim \begin{pmatrix}2&1&2\\1&3&1\\0&5&0\end{pmatrix}\begin{pmatrix}\check{v}\end{pmatrix}=\vec{0}$$

$$5\check{v}_2=0$$
$$\check{v}_2=0$$

Hence $2\check{v}_1+2\check{v}_3=0$

$$1\check{v}_1+1\check{v}_3=0$$

$$\check{v}_1=-\check{v}_3$$

let $\check{v}_3=1$ then $\check{v}_1=-1$

$$x^{(3)}t=> e^{-t}\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

$$\boxed{x(t) = c_1\begin{pmatrix}1\\1\\1\end{pmatrix}e^{4t} + c_2\begin{pmatrix}1\\-2\\1\end{pmatrix}e^{t} + c_3\begin{pmatrix}-1\\0\\1\end{pmatrix}e^{-t}}$$

18. $\mathbf{x}' = \begin{pmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ -1 & 2 & 4 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix}7\\5\\5\end{pmatrix}$

$\det(A-\lambda I)\vec{v}=0 \qquad \begin{pmatrix} -\lambda & 0 & -1 \\ 2 & -\lambda & 0 \\ -1 & 2 & 4-\lambda \end{pmatrix}\vec{v}$

$= -\lambda\left(-\lambda(4-\lambda)\right) - 1(4-\lambda) = 0 \qquad 4-(-\lambda\cdot -1)$

$4-\lambda$

$= -\lambda(-4\lambda+\lambda^2) - 4+\lambda = 0$

$4\lambda^2-\lambda^3-4-\lambda=0$

$= \lambda^3-4\lambda^2-\lambda+4=0$

$r_1 = 1$

$$\begin{array}{c|cccc} 1 & 1 & -4 & -1 & 4 \\ & & 1 & -3 & -4 \\ \hline & 1 & -3 & -4 & 0 \end{array}$$

$\lambda^2-3\lambda-4=0$

$(\lambda-4)(\lambda+1)$

Eigen Value $\Rightarrow \quad \lambda = 1, -1, 4$

Use $(A-\lambda I)\vec{v}=0$ Case $\lambda=1$ $\begin{pmatrix} -1 & 0 & -1 \\ 2 & -1 & 0 \\ -1 & 2 & 3 \end{pmatrix}\vec{v}=0$

$$-1\check{v}_1 - 1\check{v}_3 = 0$$

$$\check{v}_1 = -\check{v}_3$$

let let $\check{v}_1 = 1$ then $\check{v}_3 = -1$

$$2\check{v}_1 - \check{v}_2 = 0$$

$$-\check{v}_2 = -2$$

$$\check{v}_2 = 2$$

$$x^{(1)}t = e^{t}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

Case $\lambda = -1$

$$\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ -1 & 2 & 5 \end{pmatrix} \check{v} = 0$$

$$1\check{v}_1 - 1\check{v}_3 = 0$$

$$\check{v}_1 = \check{v}_3$$

let $\check{v}_1 = 1$ $\check{v}_3 = 1$

$$2 + \check{v}_2 = 0$$

$$\check{v}_2 = -2$$

$$x^{(2)}t = e^{-t}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Case $\lambda = 4$

$$\begin{pmatrix} -4 & 0 & -1 \\ 2 & -4 & 0 \\ -1 & 2 & 0 \end{pmatrix} \tilde{v} = 0$$

$$-4\tilde{v}_1 - \tilde{v}_3 = 0$$

$$-4\tilde{v}_1 = \tilde{v}_3$$

let $\tilde{v}_1 = 1$ then $\tilde{v}_3 = -4$

Hence $2(1) - 4\tilde{v}_2 = 0$

$$2 = 4\tilde{v}_2$$

$$\tilde{v}_2 = \frac{1}{2}$$

$$x^{(3)}t = e^{4t}\begin{pmatrix} 1 \\ \frac{1}{2} \\ -4 \end{pmatrix}$$

$$x(t) = c_1\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}e^{t} + c_2\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}e^{-t} + c_3\begin{pmatrix} 1 \\ \frac{1}{2} \\ -4 \end{pmatrix}e^{4t}$$

$$x(0) = \begin{pmatrix} 7 \\ 5 \\ 5 \end{pmatrix} \Rightarrow \begin{pmatrix} c_1 + c_2 + c_3 = 7 \\ 2c_1 - 2c_2 + \frac{1}{2}c_3 = 5 \\ -c_1 + c_2 - 4c_3 = 5 \end{pmatrix}$$

$\downarrow$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 2 & -2 & \frac{1}{2} & 5 \\ -1 & 1 & -4 & 5 \end{array}\right)$$

$$-\frac{1}{2}R_2 \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ -1 & 1 & -\frac{1}{4} & -\frac{5}{2} \\ -1 & 1 & -4 & 5 \end{array}\right)$$

$$R_2 - R_3 \left(\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ -1 & 1 & -\frac{1}{4} & -\frac{5}{2} \\ 0 & 0 & \frac{15}{4} & -\frac{15}{2} \end{array}\right)$$

$$\frac{15}{4}\vec{c}_3 = -\frac{15}{2}$$

$$\vec{c}_3 = -\frac{\cancel{15}}{\cancel{2}} \cdot \frac{\cancel{4}^{2}}{\cancel{15}}$$

$$\vec{c}_3 = -2$$

$$\frac{3}{\cancel{4}}(-2) \quad -\frac{3}{2}$$

$$7 - \frac{5}{2} = \frac{9}{2}$$

$$R_1 + R_2 \left(\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 2 & \frac{3}{4} & \frac{9}{2} \\ 0 & 0 & \frac{15}{4} & -\frac{15}{2} \end{array}\right)$$

$$\frac{12}{2} = 6$$

$$2\vec{c}_2 + \frac{3}{4}(-2) = \frac{9}{2}$$

$$2\vec{c}_2 = \frac{9}{2} + \frac{3}{2}$$

$$2\tilde{c}_2 = 6$$

$$\tilde{c}_2 = 3$$

$$1c_1 + 3 - 2 = 7$$

$$c_1 = 7 - 1$$

$$c_1 = 6$$

$$\boxed{x(t) = 6\begin{pmatrix}1\\2\\-1\end{pmatrix}e^{t} + 3\begin{pmatrix}1\\-2\\1\end{pmatrix}e^{-t} - 2\begin{pmatrix}1\\ \frac{1}{2}\\-4\end{pmatrix}e^{4t}}$$

In each of Problems 24 through 27, the eigenvalues and eigenvectors of a matrix **A** are given. Consider the corresponding system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

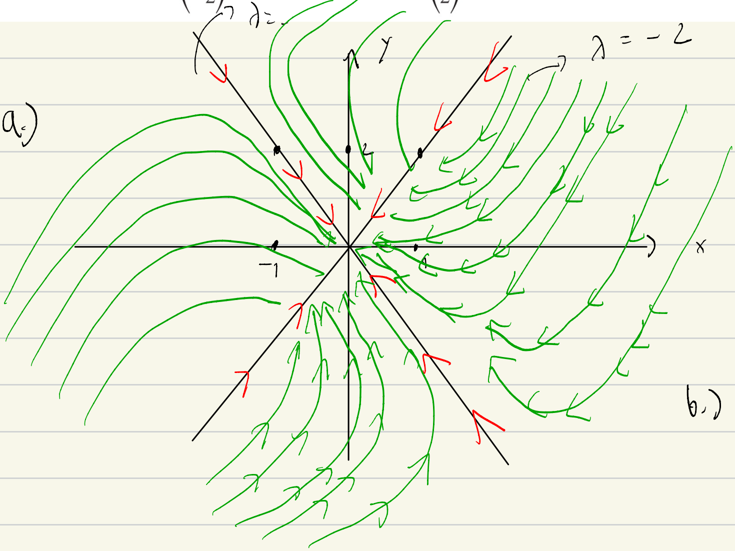
(a) Sketch a phase portrait of the system.

(b) Sketch the trajectory passing through the initial point $(2, 3)$.

(c) For the trajectory in part (b), sketch the graphs of x_1 versus t and of x_2 versus t on the same set of axes.

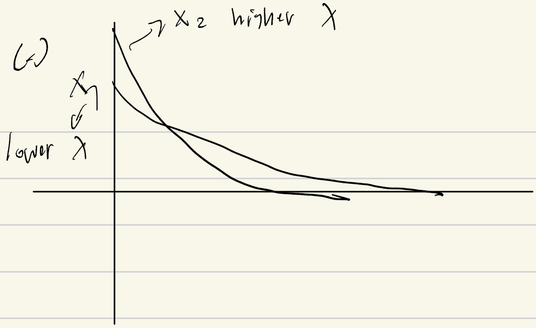
24. $r_1 = -1, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix}-1\\2\end{pmatrix}; \qquad r_2 = -2, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix}1\\2\end{pmatrix}$

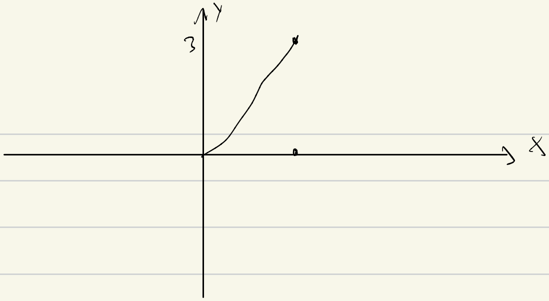
a.)



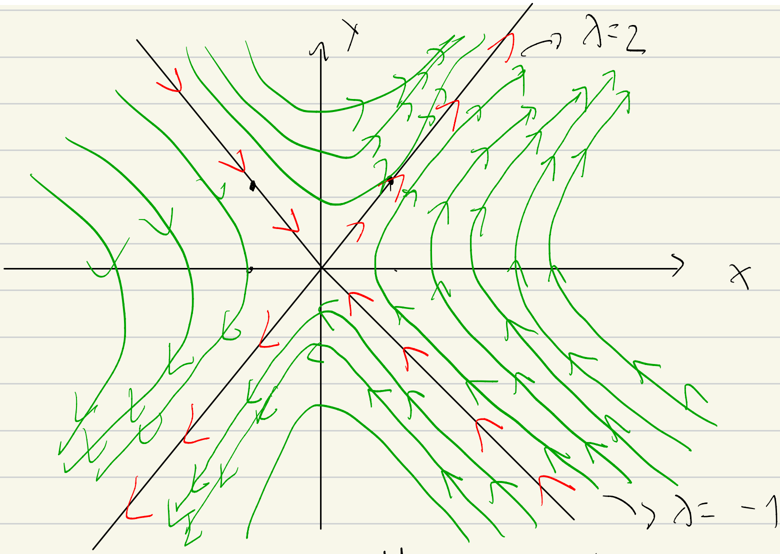
b.) Pass through $(2, 3)$

(c)





26. $r_1 = -1, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}; \qquad r_2 = 2, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$



saddle unstable

28. Consider a 2×2 system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. If we assume that $r_1 \neq r_2$, the general solution is $\mathbf{x} = c_1\boldsymbol{\xi}^{(1)}e^{r_1 t} + c_2\boldsymbol{\xi}^{(2)}e^{r_2 t}$, provided that $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ are linearly independent. In this problem we establish the linear independence of $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ by assuming that they are linearly dependent and then showing that this leads to a contradiction.

(a) Note that $\boldsymbol{\xi}^{(1)}$ satisfies the matrix equation $(\mathbf{A} - r_1\mathbf{I})\boldsymbol{\xi}^{(1)} = \mathbf{0}$; similarly, note that $(\mathbf{A} - r_2\mathbf{I})\boldsymbol{\xi}^{(2)} = \mathbf{0}$.

(b) Show that $(\mathbf{A} - r_2\mathbf{I})\boldsymbol{\xi}^{(1)} = (r_1 - r_2)\boldsymbol{\xi}^{(1)}$.

(c) Suppose that $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ are linearly dependent. Then $c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)} = \mathbf{0}$ and at least one of c_1 and c_2 (say c_1) is not zero. Show that $(\mathbf{A} - r_2\mathbf{I})(c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)}) = \mathbf{0}$, and also show that $(\mathbf{A} - r_2\mathbf{I})(c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)}) = c_1(r_1 - r_2)\boldsymbol{\xi}^{(1)}$. Hence $c_1 = 0$, which is a contradiction. Therefore, $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ are linearly independent.

(d) Modify the argument of part (c) if we assume that $c_2 \neq 0$.

(e) Carry out a similar argument for the case in which the order n is equal to 3; note that the procedure can be extended to an arbitrary value of n.

(b.) $(A - r_2 I)\xi^{(1)} = A\xi^{(1)} - r_2 I \xi^{(1)}$

as $A\xi^{(1)} = r_1\xi^{(1)}$

then $r_1\xi^{(1)} - r_2\xi^{(1)} \Rightarrow (r_1 - r_2)\xi^{(1)}$ # Prove

(c.) ↳ for first one => Assume that $c_1 \neq 0$ for linearly dependent then it is clear that $(A - r_2 I)(c_1\xi^{1} + c_2\xi^{(2)}) = 0$ → linear dependent; there exist a constant such that $c_1\vec{v}_1 + c_2\vec{v}_2 =$ and either c_1 or $c_2 \neq 0$

$$(A - r_2 I)(c_1\xi^{(1)} + c_2\xi^{(2)}) = 0$$

$$Ac_1\xi^{(1)} + Ac_2\xi^{(2)} - r_2 c_1\xi^{(1)} - r_2 c_2\xi^{(2)} = 0$$

$$c_1 r_1\xi^{(1)} + \cancel{r_2 c_2\xi^{(2)}} - r_2 c_1\xi^{(1)} - \cancel{r_2 c_2\xi^{(2)}} = 0$$

$$\boxed{c_1(r_1 - r_2)\xi^{(1)} = 0}$$

$$c_1 = 0$$

29. Consider the equation

$$ay'' + by' + cy = 0, \qquad \text{(i)}$$

where a, b, and c are constants with $a \neq 0$. In Chapter 3 it was shown that the general solution depended on the roots of the characteristic equation

$$ar^2 + br + c = 0. \qquad \text{(ii)}$$

(a) Transform Eq. (i) into a system of first order equations by letting $x_1 = y, x_2 = y'$. Find the system of equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$ satisfied by $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

(b) Find the equation that determines the eigenvalues of the coefficient matrix $\mathbf{A}$ in part (a). Note that this equation is just the characteristic equation (ii) of Eq. (i).

(a.) => if $x_1 = y$ and $x_2 = y'$ then $y'' = x_2'$ also $x_1' = x_2$

$$a x_2' + b x_2 + c x_1 = 0$$

$$a x_2' = -c x_1 - b x_2$$

as $x_1' = x_2$

$$x_2' = \frac{-c}{a} x_1 - \frac{b}{a} x_2$$

$$= 0 x_1 + x_2$$

$$- \frac{c}{a} x_1 - \frac{b}{a} x_2$$

$$x' = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix} x$$

Hence System of equation is

$$x_1' = x_2$$

$$x_2' = -\frac{c}{a} x_1 - \frac{b}{a} x_2$$

b.) As the matrix is $x' = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix}$

Hence equation is $\det(A - rI) = 0$

$$x' = \begin{pmatrix} -r & 1 \\ -\frac{c}{a} & -\frac{b}{a} - r \end{pmatrix}$$

$= \text{give} = \frac{b}{a} r + r^2 + \frac{c}{a} = 0$

$= r^2 + \frac{b}{a} r + \frac{c}{a} = 0$

or $ar^2 + br + c = 0$

r

Prove

30. The two-tank system of Problem 22 in Section 7.1 leads to the initial value problem

$$\mathbf{x}' = \begin{pmatrix} -\frac{1}{10} & \frac{3}{40} \\ \frac{1}{10} & -\frac{1}{5} \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} -17 \\ -21 \end{pmatrix},$$

where x_1 and x_2 are the deviations of the salt levels Q_1 and Q_2 from their respective equilibria.

(a) Find the solution of the given initial value problem.

(b) Plot x_1 versus t and x_2 versus t on the same set of axes.

(c) Find the smallest time T such that $|x_1(t)| \leq 0.5$ and $|x_2(t)| \leq 0.5$ for all $t \geq T$.

31. Consider the system

$-\frac{1}{10} - \frac{1}{5}$

$-\frac{3}{10}$

$\lambda^2 + \frac{3}{10}\lambda + \frac{1}{80} = 0$

$= 80\lambda^2 + 24\lambda + 1 = 0$

$$\frac{-24 \pm \sqrt{(24)^2 - 4(80)(1)}}{160}$$

$$= \frac{-24 \pm \sqrt{256}}{160} \rightarrow 16$$

$$= \frac{-24 \pm 16}{160} = -\frac{3}{20} \pm \frac{1}{10}$$

r

Case $\lambda = -\frac{3}{20} + \frac{1}{10} = -\frac{1}{20}$

$-\frac{1}{10} + \frac{1}{\ }$.

$$(A - \lambda I)\check{v} = \begin{pmatrix} -\frac{1}{20} & \frac{3}{40} \\ \frac{1}{10} & -\frac{3}{20} \end{pmatrix} \check{v} = 0$$

$$-\frac{1}{20}\check{v}_1 + \frac{3}{40}\check{v}_2 = 0$$

$$\frac{3}{40}\check{v}_2 = \frac{1}{20}\check{v}_1$$

If $\check{v}_1 = 3$

then $\check{v}_2 = \frac{\cancel{3}}{\cancel{20}} \times \frac{\cancel{40}^{2}}{\cancel{3}}$ $\check{v}_2 = 2$

then $x^{(1)}t = e^{-\frac{t}{20}}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$-4+10$ $\frac{6}{40}$

$-\frac{1}{20} + \frac{1}{4}$

Case $\lambda = \frac{-3}{20} - \frac{1}{10} \Rightarrow \frac{-5}{20} = -\frac{1}{4}$

$-\frac{1}{5} + \frac{1}{4} = -\frac{4}{20} + \frac{5}{20}$

$= \frac{1}{20}$

$$\begin{pmatrix} \frac{6}{40} & \frac{3}{40} \\ \frac{1}{10} & \frac{1}{20} \end{pmatrix} \check{v} = 0$$

$$\frac{1}{10}\check{v}_1 + \frac{1}{20}\check{v}_2 = 0$$

If $\check{v}_1 = 1$ and let $\check{v}_2 = -2$

$$\boxed{x(t) = c_1\begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-\frac{t}{20}} + c_2\begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-\frac{1}{4}t}}$$

$$x(0) = \begin{pmatrix} -17 \\ -21 \end{pmatrix} \Rightarrow c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{matrix} -17 \\ -21 \end{matrix}$$

$$3c_1 + c_2 = -17$$
$$2c_1 - 2c_2 = -21$$

$$6c_1 + 2c_2 = -34$$
$$2c_1 - 2c_2 = -21$$

$$8c_1 = -55$$

17x
2
2

$$c_1 = \frac{-55}{8}$$

$2\left(\frac{-55}{8_4}\right) = -$

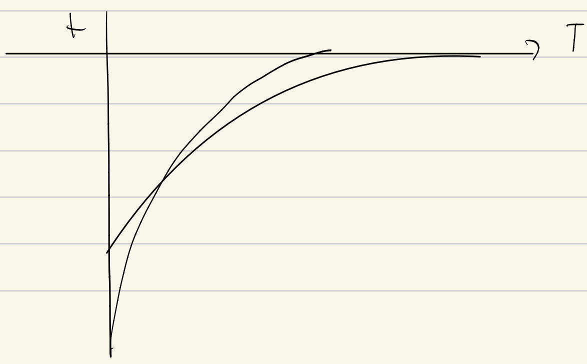
-27+

$$-21 + \frac{55}{4}$$

$$\frac{-84+55}{4} = \frac{-29}{4}$$ 86

$$-2c_2 = \frac{-29}{4} \rightarrow c_2 = \frac{+29}{8}$$

$$\boxed{\text{Hence } x(t) = -\frac{55}{8}\begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-\frac{t}{20}} + \frac{29}{8}\begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-\frac{1}{4}t}}$$



31. Consider the system

$$\mathbf{x}' = \begin{pmatrix} -1 & -1 \\ -\alpha & -1 \end{pmatrix} \mathbf{x}.$$

(a) Solve the system for $\alpha = 0.5$. What are the eigenvalues of the coefficient matrix? Classify the equilibrium point at the origin as to type.

(b) Solve the system for $\alpha = 2$. What are the eigenvalues of the coefficient matrix? Classify the equilibrium point at the origin as to type.

(c) In parts (a) and (b), solutions of the system exhibit two quite different types of behavior. Find the eigenvalues of the coefficient matrix in terms of α, and determine the value of α between 0.5 and 2 where the transition from one type of behavior to the other occurs.

(a.) => at $\alpha = 0.5$ $\quad x' = \begin{pmatrix} -1 & -1 \\ -0.5 & -1 \end{pmatrix} x$

$1 - (-0.5 \times -1)$

$1 - (0.5$

$1 - 0.5$

Eigen Value => $\det(A - \lambda I) = 0$

$$= \lambda^2 + 2\lambda + \frac{1}{2} = 0$$

$$2\lambda^2 + 4\lambda + 1 = 0$$

$$\frac{-4 \pm \sqrt{16 - 4(2)(1)}}{4} = \frac{-4 \pm \sqrt{8}}{4}$$

$$= -1 \pm \frac{\sqrt{2}}{2}$$

Eigen Value => $-1 + \frac{\sqrt{2}}{2}$, $-1 - \frac{\sqrt{2}}{2}$

both negative → nodal sink (Stable)

(b.) $\quad x' = \begin{pmatrix} -1 & -1 \\ -2 & -1 \end{pmatrix} X$

$1 - (-2 \times -1) = 1 - 2 = -1$

$$= \lambda^2 + 2\lambda - 1 = 0$$

$$= \frac{-2 \pm \sqrt{4 - (4)(1)(-1)}}{2} = \frac{-2 \pm \sqrt{4 + 4}}{2} = \frac{-2 \pm \sqrt{8}}{2}$$

$$\lambda = -1 \pm \sqrt{2}$$

$$\lambda_1 > 0, \quad \lambda_2 < 0$$

Saddle point

(c.) In term of alpha $\longrightarrow$ $x' = \begin{pmatrix} -1 & -1 \\ -\alpha & -1 \end{pmatrix} x$ $\quad 1-(-\alpha \times -1) = 1-\alpha$

$= \lambda^2 + 2\lambda + (1-\alpha) = 0$

$\dfrac{-2 \pm \sqrt{4-4(1)(1-\alpha)}}{2}$

$= \dfrac{-2 \pm \sqrt{4-4+4\alpha}}{2}$

$\lambda = -1 \pm \sqrt{\alpha}$

between 0.5 to 1

It is still $\lambda_1 \lambda_2 < 0 \rightarrow$ Nodal sink (Stable)

$-1+1=0 \rightarrow$ Positive

$-1-1=-2 \rightarrow$ negativ

as alpha approach to 1 it changes from Nodal sink to saddle as increase

33. Consider the preceding system of differential equations (i).

(a) Find a condition on R_1, R_2, C, and L that must be satisfied if the eigenvalues of the coefficient matrix are to be real and different.

(b) If the condition found in part (a) is satisfied, show that both eigenvalues are negative. Then show that $I(t) \to 0$ and $V(t) \to 0$ as $t \to \infty$, regardless of the initial conditions.

(c) If the condition found in part (a) is not satisfied, then the eigenvalues are either complex or repeated. Do you think that $I(t) \to 0$ and $V(t) \to 0$ as $t \to \infty$ in these cases as well?

Hint: In part (c), one approach is to change the system (i) into a single second order equation. We also discuss complex and repeated eigenvalues in Sections 7.6 and 7.8.

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} -\dfrac{R_1}{L} & -\dfrac{1}{L} \\ \dfrac{1}{C} & -\dfrac{1}{CR_2} \end{pmatrix} \begin{pmatrix} I \\ V \end{pmatrix}.$$

$= \lambda^2 + \left(\dfrac{R_1}{L} + \dfrac{1}{CR_2}\right)\lambda + \left(\dfrac{R_1}{LCR_2} + \dfrac{1}{CL}\right)$

for Eigen Value to be real then $\quad b^2 - 4ac > 0$

$\left(\dfrac{R_1}{L} + \dfrac{1}{CR_2}\right)^2 - 4(1)\left(\dfrac{R_1}{LCR_2} + \dfrac{1}{CL}\right) > 0$

$\dfrac{R_1^2}{L^2} + \dfrac{2R_1}{LCR_2} + \dfrac{1}{(CR_2)^2} - \dfrac{4R_1}{LCR_2} - \dfrac{4}{CL} > 0$

$= \boxed{\dfrac{R_1^2}{L^2} - \dfrac{2R_1}{LCR_2} + \dfrac{1}{C^2R_2^2} - \dfrac{4}{CL} > 0}$

(b.) from Characteristic eq $\Rightarrow \lambda^2 - Tr(A)\lambda + Det(A) = 0$

We Know $\lambda_1 + \lambda_2 = Tr(A)$ and $\lambda_1 \cdot \lambda_2 = Det(A)$

$Tr(A) = \left(\dfrac{-R_1}{L} - \dfrac{1}{CR_2}\right)$ negative ; $Det(A) = \left(\dfrac{R_1}{L} \times \dfrac{1}{CR_2} + \dfrac{1}{CL}\right)$ Positive

Hence

$-\lambda_1 + (-\lambda_2) =$ negative $\quad -\lambda_1 \cdot -\lambda_2 =$ Positive

As $\lambda_1 < \lambda_2 < 0 \rightarrow$ It is nodal sink; Hence Stable at (0,0).

(c.) Yes ; As long as long as all the parameter are positive ; It will converge to zero

7.6 Complex Eigen Value

$-2-(-\frac{9}{2}) \Rightarrow -2+\frac{9}{2} = \frac{5}{2}$

4. $\mathbf{x}' = \begin{pmatrix} 2 & -\frac{5}{2} \\ \frac{9}{5} & -1 \end{pmatrix} \mathbf{x}$ $\longrightarrow$ $\lambda^2 - \lambda + \frac{5}{2} = 0$

$= 2\lambda^2 - 2\lambda + 5 = 0$

$\dfrac{2 \pm \sqrt{4-4(2)(5)}}{4}$

$= \dfrac{2 \pm \sqrt{4-40}}{4}$

$= \dfrac{2 \pm \sqrt{-36}}{4} \Rightarrow \dfrac{2 \pm 6i}{4} \Rightarrow \frac{1}{2} \pm \frac{3}{2}i$

Eigen Value

Eigen Vector $\Rightarrow$ use $\left(\frac{1}{2} + \frac{3}{2}i\right)$

$$(A - \lambda I)\vec{v} = 0 \Rightarrow \begin{pmatrix} 2 - \frac{1}{2} - \frac{3}{2}i & -\frac{5}{2} \\ \frac{9}{5} & -1 - \frac{1}{2} - \frac{3}{2}i \end{pmatrix} (\vec{v}) = \vec{0}$$

$$= \begin{pmatrix} \frac{3}{2} - \frac{3}{2}i & -\frac{5}{2} \\ \frac{9}{5} & -\frac{3}{2} - \frac{3}{2}i \end{pmatrix} \vec{v} = 0$$

$$\left(\frac{3}{2} - \frac{3}{2}i\right)\vec{v}_1 - \frac{5}{2}\vec{v}_2 = 0$$

let $\vec{v}_1 = 5$ then $\dfrac{15-15i}{\cancel{2}} = \dfrac{5}{\cancel{2}}\vec{v}_2$

$\vec{v}_2 = 3-3i$

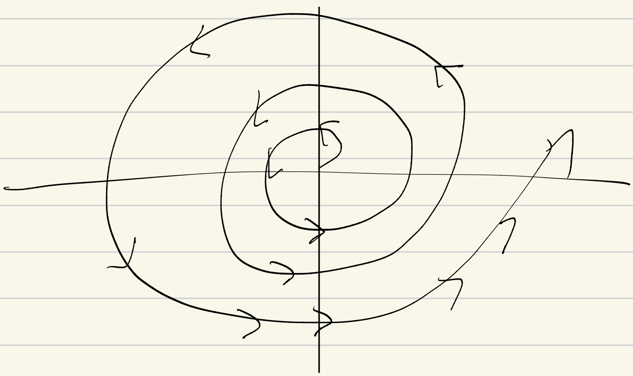
$\vec{v}_1 = 5$

$$x(t) = e^{\frac{1}{2}t}\left(\cos\left(\tfrac{3}{2}t\right) + i\sin\left(\tfrac{3}{2}t\right)\right)\begin{pmatrix} 5 \\ 3-3i \end{pmatrix}$$

$$\Rightarrow e^{\frac{t}{2}}\begin{pmatrix} 5\cos\left(\frac{3}{2}t\right) + 5i\sin\left(\frac{3}{2}t\right) \\ 3\cos\left(\frac{3}{2}t\right) - 3i\cos\left(\frac{3}{2}t\right) + 3i\sin\left(\frac{3}{2}t\right) + 3\sin\left(\frac{3}{2}t\right) \end{pmatrix}$$

$$x(t) = c_1 e^{\frac{t}{2}}\begin{pmatrix} 5\cos\left(\frac{3}{2}t\right) \\ 3\cos\left(\frac{3}{2}t\right) + 3\sin\left(\frac{3}{2}t\right) \end{pmatrix} + c_2 e^{\frac{t}{2}}\begin{pmatrix} 5\sin\left(\frac{3}{2}t\right) \\ -3\cos\left(\frac{3}{2}t\right) + 3\sin\left(\frac{3}{2}t\right) \end{pmatrix}$$

as the real part are positive then it will Spiral outward.



Choose point $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & -\frac{5}{2} \\ \frac{9}{5} & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} -\frac{5}{2} \\ -1 \end{pmatrix}$ Counter Clockwise

8. $\mathbf{x}' = \begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix} \mathbf{x}$

$= \operatorname{Det}(A - \lambda I) = 0$

$$\begin{pmatrix} -3-\lambda & 0 & 2 \\ 1 & -1-\lambda & 0 \\ -2 & -1 & -\lambda \end{pmatrix} \vec{v} = 0$$

$$= (-3-\lambda)(-\lambda(-1-\lambda)) + 2(-1+2(-1-\lambda)) = 0$$

$$(-3-\lambda)(\lambda+\lambda^2) + 2(-1-2-2\lambda) = 0$$

$$\cancel{-3}\lambda - \cancel{3\lambda^2} - \cancel{\lambda^2} - \cancel{\lambda^3} + 2(-3-2\lambda) = 0$$

$$-6 - \cancel{4}\lambda$$

$$-\lambda^3 - 4\lambda^2 - 7\lambda - 6 = 0$$

$$\lambda^3 + 4\lambda^2 + 7\lambda + 6 = 0$$

1 root $= -2$

$$\begin{array}{r|rrrr} -2 & 1 & 4 & 7 & 6 \\ & & -2 & -4 & -6 \\ \hline & 1 & 2 & 3 & 0 \end{array}$$

$$\lambda^2 + 2\lambda + 3 = 0$$

$$\frac{-2 \pm \sqrt{(2)^2 - 4(1)(3)}}{2}$$

$$\frac{\sqrt{4}\sqrt{2}}{2} = \frac{2\sqrt{2}}{2}$$

$$= \frac{-2 \pm \sqrt{4 - 12}}{2} \Rightarrow \frac{-2 \pm \sqrt{-8}}{2}$$

$$= -1 \pm \sqrt{2}\, i$$

root $= -2, -1 \pm \sqrt{2}\, i$

Eigen Vector case -2

$$(A - \lambda I)\check{v} \Rightarrow \begin{pmatrix} -1 & 0 & 2 \\ 1 & 1 & 0 \\ -2 & -1 & 2 \end{pmatrix} \vec{v} = \vec{0}$$

$$-1\check{v}_1 + 2\check{v}_3 = 0$$

$$2\check{v}_3 = \check{v}_1$$

let $\check{v}_3 = 1$ then $\check{v}_1 = 2$

$$2 + \check{v}_2 = 0$$

$$\check{v}_2 = -2$$

$$x^{(1)}t = e^{-2t}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

case $-1+\sqrt{2}i$ $\begin{pmatrix} -2-\sqrt{2}i & 0 & 2 \\ 1 & -\sqrt{2}i & 0 \\ -2 & -1 & 1-\sqrt{2}i \end{pmatrix} \check{v} = 0$

$\check{v}_1 + \sqrt{2}i\check{v}_2 = 0$ $\check{v}_1 = \sqrt{2}i\check{v}_2$

let $\check{v}_2 = 1$

let $\check{v}_2 = 1$ $\check{v}_1 = -\sqrt{2}i$

$\check{v}_1 = \sqrt{2}i$

Hence $(-2+\sqrt{2}i)(-\sqrt{2}i) + 2\check{v}_3 = 0$

$2\sqrt{2}i - 2i^2 + 2\check{v}_3 = 0$ $(-2-\sqrt{2}i)\sqrt{2}i + 2\check{v}_3 = 0$

$2\sqrt{2}i + 2 + 2\check{v}_3 = 0$ $-2\sqrt{2}i + 2 = -2\check{v}_3$

$\check{v}_3 = -\sqrt{2}i - 1$

case $\lambda = -1+\sqrt{2}i$

$\check{v}_3 = \sqrt{2}i - 1$

$$x^{(2)}t = e^{-t}(\cos(\sqrt{2}t) + i\sin(\sqrt{2}t))\begin{pmatrix} \sqrt{2}i \\ 1 \\ \sqrt{2}i - 1 \end{pmatrix}$$

$$= e^{-t}\begin{pmatrix} \sqrt{2}i\cos(\sqrt{2}t) - \sqrt{2}\sin(\sqrt{2}t) \\ +\cos(\sqrt{2}t) \quad + i\sin(\sqrt{2}t) \\ \sqrt{2}i\cos(\sqrt{2}t) - \cos(\sqrt{2}t) - \sqrt{2}\sin(\sqrt{2}t) - i\sin(\sqrt{2}t) \end{pmatrix}$$

$$= e^{-t}\begin{pmatrix} -\sqrt{2}\sin(\sqrt{2}t) \\ +\cos(\sqrt{2}t) \\ -\cos(\sqrt{2}t) - \sqrt{2}\sin(\sqrt{2}t) \end{pmatrix} + ie^{-t}\begin{pmatrix} \sqrt{2}\cos\sqrt{2}t \\ \sin(\sqrt{2}t) \\ \sqrt{2}\cos(\sqrt{2}t) \quad i\sin(\sqrt{2}t) \end{pmatrix}$$

9. $\mathbf{x}' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Find Sol due to initial Problem.

$$\det(A - \lambda I) = 0 \Rightarrow \lambda^2 + 2\lambda + 2 = 0$$

$$\frac{-2 \pm \sqrt{4 - 4(1)(2)}}{2}$$

$$= \frac{-2 \pm \sqrt{-4}}{2}$$

$$\frac{-2}{2} \pm \frac{\sqrt{4}\sqrt{-1}}{2}$$

$$= -1 \pm i$$

use $\lambda = -1 + i$ $\qquad (A - \lambda I)\vec{v} = 0$

$$\begin{pmatrix} 1 - (-1 + i) & -5 \\ 1 & -3 - (-1 + i) \end{pmatrix} \vec{v} = 0$$

$$= \begin{pmatrix} 2 - i & -5 \\ 1 & -2 - i \end{pmatrix} \vec{v} = 0$$

$$1\vec{v}_1 + (-2 - i)\vec{v}_2 = 0$$

$4i - 2 + 2 + i$

let $\vec{v}_1 = (2i - 1)$ and $\vec{v}_2 = i$

$(2 - i)(2i - 1)$ $\quad -5i$

Hence $x^{(1)}t = e^{-t}(\cos(t) + i\sin(t))\begin{pmatrix} 2i - 1 \\ i \end{pmatrix}$

$$= e^{-t}\begin{pmatrix} 2i\cos(t) - \cos(t) - 2\sin(t) - i\sin(t) \\ i\cos(t) - \sin(t) \end{pmatrix}$$

$$= e^{-t}\begin{pmatrix} -\cos(t) - 2\sin(t) \\ -\sin(t) \end{pmatrix} + ie^{-t}\begin{pmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{pmatrix}$$

$$x(t) = c_1 e^{-t}\begin{pmatrix} -\cos(t) - 2\sin(t) \\ -\sin(t) \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{pmatrix}$$

Initial con $x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad c_1\begin{pmatrix} -1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{matrix} 1 \\ 1 \end{matrix}$

$$-c_1 + 2c_2 = 1$$

$$c_2 = 1$$

$$-c_1 + 2 = 1$$

$$-c_1 = 1 - 2$$

$$-c_1 = -1$$

$$c_1 = 1$$

10. $\mathbf{x}' = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$3 - (-2) = 5$

$= \lambda^2 + 4\lambda + 5 = 0$

$$\frac{-4 \pm \sqrt{(4)^2 - 4(1)(5)}}{2} = \frac{-4 \pm \sqrt{16 - 20}}{2} = \frac{-4 \pm \sqrt{-4}}{2}$$

$= -2 \pm i$

Ws $\lambda = -2 + i$

$$\Rightarrow \begin{pmatrix} -3-(-2+i) & 2 \\ -1 & -1-(-2+i) \end{pmatrix} \vec{v} = 0$$

$$= \begin{pmatrix} -1-i & 2 \\ -1 & 1-i \end{pmatrix} \vec{v} = 0$$

$-1\vec{v}_1 + (1-i)\vec{v}_2 = 0$

$\vec{v}_1 = (1-i)\vec{v}_2$

let $\vec{v}_2 = 1$ then $\vec{v}_1 = 1 - i$

$$x(t) = e^{-2t}(\cos(t) + i\sin(t)) \begin{pmatrix} 1-i \\ 1 \end{pmatrix}$$

$$= e^{-2t} \begin{pmatrix} \cos(t) - i\cos(t) + i\sin(t) + \sin(t) \\ \cos(t) + i\sin(t) \end{pmatrix}$$

$$= e^{-2t}\begin{pmatrix} \cos(t) + \sin(t) \\ \cos(t) \end{pmatrix} + ie^{-2t}\begin{pmatrix} -\cos(t) + \sin(t) \\ \sin(t) \end{pmatrix}$$

General Sol $\Rightarrow$ $x(t) \Rightarrow c_1\begin{pmatrix} \cos(t) + \sin(t) \\ \cos(t) \end{pmatrix} e^{-2t} + c_2\begin{pmatrix} -\cos(t) + \sin(t) \\ \sin(t) \end{pmatrix}$

$-1 \quad \begin{bmatrix} 1 & e^{-2t} \\ -2 \end{bmatrix}$

28. A mass m on a spring with constant k satisfies the differential equation (see Section 3.7)

$$mu'' + ku = 0,$$

where $u(t)$ is the displacement at time t of the mass from its equilibrium position.
(a) Let $x_1 = u, x_2 = u'$, and show that the resulting system is

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -k/m & 0 \end{pmatrix}\mathbf{x}.$$

(b) Find the eigenvalues of the matrix for the system in part (a).
(c) Sketch several trajectories of the system. Choose one of your trajectories, and sketch the corresponding graphs of x_1 versus t and x_2 versus t. Sketch both graphs on one set of axes.
(d) What is the relation between the eigenvalues of the coefficient matrix and the natural frequency of the spring–mass system?

(a) as $x_1 = u$ and $x_2 = u'$

then Also $x_1' = x_2$ and $u'' = x_2'$

Hence $mx_2' + kx_1 = 0$

$$mx_2' = -kx_1$$

$$x_2' = -\frac{k}{m}x_1$$

2 system ODE. $\rightarrow$ as $x_1' = x_2$

$$x_2' = -\frac{k}{m}x_1$$

give $x_1' = 0x_1 + x_2$

$$x_2' = -\frac{k}{m}x_1 + 0x_2$$

$$= \begin{pmatrix} 0 & 1 \\ -\frac{K}{M} & 0 \end{pmatrix} X$$

Sastify $X' = \begin{pmatrix} 0 & 1 \\ -\frac{K}{M} & 0 \end{pmatrix} X$

Eigen Value $\Rightarrow$ $\lambda^2 + \frac{K}{M} = 0$

$$\lambda^2 = -\frac{K}{M}$$

$$\lambda = \pm\sqrt{-\frac{K}{M}}$$

$$\lambda = \pm\sqrt{\frac{K}{M}}\, i$$

Purely Imaginary = Centre.

Chapter 9

6. $dx/dt = 1 + 2y, \qquad dy/dt = 1 - 3x^2$

$x' = 1 + 2y \quad ; \quad y' = 1 - 3x^2$

Crit point $\Rightarrow$ $x'(t) = 0, \; y'(t) = 0$

$1 + 2y = 0$,

$1 - 3x^2 = 0$ $\qquad x = \pm \frac{1}{\sqrt{3}}$

$2y = -1$

$y = -\frac{1}{2}$ $\longrightarrow$ Point = $\left(\frac{1}{\sqrt{3}}, -\frac{1}{2}\right)$ $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{2}\right)$

Crit Point 1 $\qquad$ Crit Point 2

$$x' = 1 + 2y$$
$$y' = 1 - 3x^2$$

use linearisation

$$\begin{vmatrix} 0 & 2 \\ -6x & 0 \end{vmatrix}$$

At $\left(\frac{1}{\sqrt{3}}, -\frac{1}{2}\right) \rightarrow \begin{vmatrix} 0 & 2 \\ -\frac{6}{\sqrt{3}} & 0 \end{vmatrix}$

$$\lambda^2 + \frac{12}{\sqrt{3}} = 0$$

$$\lambda^2 = -\frac{12}{\sqrt{3}}$$

$$\lambda = \pm \frac{\sqrt{12}}{3} i$$

no real part Hence it is Center

stable → Periodic Cycle

at $\left(\frac{-1}{\sqrt{3}}, \frac{-1}{2}\right) \rightarrow \begin{vmatrix} 0 & 2 \\ \frac{6}{\sqrt{3}} & 0 \end{vmatrix}$

$$= \lambda^2 - \frac{12}{\sqrt{3}} = 0$$

$$\lambda^2 = \frac{12}{\sqrt{3}}$$

$$x = \pm \frac{\sqrt{12}}{3}$$

as $\lambda_1 > 0$ and $\lambda_2 < 0$

Hence it is Saddle Point and Unstable.

7. $dx/dt = 2x - x^2 - xy, \qquad dy/dt = 3y - 2y^2 - 3xy$

15. $dx/dt = x(2 - x - y), \qquad dy/dt = -x + 3y - 2xy$

$x'(t) = x(2 - x - y)$

$y'(t) = -x + 3y - 2xy$

at $x = 0 \rightarrow y = 0$

as $2 - x - y = 0$

$y = 2 - x$

sub to eq 2: $-x + 3(2-x) - 2x(2-x) = 0$

$-x + 6 - 3x - 4x + 2x^2 = 0$

$2x^2 - 8x + 6 = 0$

$x^2 - 4x + 3 = 0$

$(x-3)(x-1) = 0$

$x = 3, \ x = 1$

at $x = 3 \rightarrow y = -1$, at $x = 1 \rightarrow y = 1$

<u>Crit Point</u> $\Rightarrow (0,0) \ (3,-1) \ (1,1)$

<u>linearisation</u> $\rightarrow \begin{vmatrix} \frac{\partial f}{\partial x} x'_1 & \frac{\partial f}{\partial y} x'_1 \\ \frac{\partial g}{\partial x} y_1 & \frac{\partial g}{\partial y} y'_1 \end{vmatrix} = \begin{vmatrix} 2-2x-y & -x \\ -1-2y & 3-2x \end{vmatrix}$ $-1-(3) =$

$t(1,1) = \begin{vmatrix} -1 & -1 \\ -3 & 1 \end{vmatrix}$ $\lambda^2 - 4 = 0$

$\lambda^2 = 4 \rightarrow \lambda = \pm 2$

as $\lambda_1 < 0$ and $\lambda_2 > 0$ it is Unstable

10. $dx/dt = (2+x)(y-x), \qquad dy/dt = y(2+x-x^2)$

$x'(t) = (2+x)(y-x) \rightarrow 2y - 2x + xy - x^2$

$y'(t) = y(2+x-x^2) \rightarrow 2y + xy - x^2 y$

$2+x=0 \rightarrow x=-2 \rightarrow y=0$

$y=0$ $\quad (-2,0) \rightarrow$ 1st

$y-x=0$

$y=x$

$y=0 \rightarrow x=0 \rightarrow (0,0) \rightarrow$ 2nd

$y-x=0 \rightarrow y=x$

$2+x-x^2=0$

$x^2-x-2=0$

$(x-2)(x+1)$

$\downarrow$

$x=2,-1 \rightarrow (2,2) \quad (-1,-1)$

$\qquad\qquad\qquad\qquad\quad\ \downarrow \qquad\quad \downarrow$

$\qquad\qquad\qquad\qquad\quad\ 3 \qquad\quad\ 4$

Check Nature of equilibrium.

linearisation $\Rightarrow \begin{vmatrix} -2+y-2x & 2+x \\ y-2xy & 2+x-x^2 \end{vmatrix}$

at $(0,0) \Rightarrow \begin{vmatrix} -2 & 2 \\ 0 & 2 \end{vmatrix} \Rightarrow \lambda^2-4=0$

$\lambda^2=4$

$\downarrow$

$\lambda = \pm 2$

$\downarrow$

$\lambda_1 < 0 < \lambda_2$

$\downarrow$

Saddle point

at $(3,2) \Rightarrow \begin{vmatrix} -4 & 4 \\ -6 & 0 \end{vmatrix}$

$\lambda^2 + 4\lambda + 24 = 0$

$$\frac{-4 \pm \sqrt{16 - 4(1)(24)}}{2a}$$

$2 - 8$

$$= -2 \pm \frac{\sqrt{16 - 4(24)}}{2}$$

negative

Complex

Spiral Sink

as the real part is negative; then it it is asymptically stable