$x = 1 - t$ $\quad$ $t = \frac{x-1}{-1}, t = \frac{y-2}{1},$

$t = 1 - x$

$-3t = z$ $\quad$ $t = \frac{z}{-3}$ $\quad$ $y = t + 2$ $\quad$ A

$y - 2 = t$

1. Find the equation of the plane that passes through the point $(1, 0, 2)$ and contains the line whose vector eq is $r(t) = \langle 1-t, t+2, -3t \rangle$.

$r(t) = \langle 1-t, t+2, -3t \rangle$ $\quad \rightarrow x = x_0 + t$

$\rightarrow$ point $\overset{B}{(1, 2, 0)}$ and vector $\langle -1, 1, -3 \rangle$

so $\vec{AB} = \langle 0, 2, -2 \rangle$ $\quad$ $\vec{r(t)} = \langle -1, 1, -3 \rangle$

Vector that orthogonal to the plane (normal vector $= \vec{AB} \times \vec{r(t)}$

$$= \left| \begin{matrix} i & j & k \\ 0 & 2 & -2 \\ -1 & 1 & -3 \end{matrix} \right| \begin{matrix} i & j \\ 0 & 2 \\ -1 & 1 \end{matrix} = -6i + 2j + 0k - (-2k - 2i + 0j)$$

$$= -4i + 2j + 2k.$$

eq of the plane $= -4(x-1) + 2(y-0) + 2(z-2) = 0$

$= -4x + 4 + 2y + 2z - 4 = 0$

$-4x + 2y + 2z = 0$

#

2. (a) *Find the point of intersection of the curves*

$$\mathbf{r}(t) = \langle 1-t, t+2, -3t\rangle \quad \textit{and} \quad \mathbf{s}(u) = \langle u, 2u, u^2-u\rangle.$$

(b) *Find the acute angle between the tangent lines to these two curves at the point of intersection. (You may give your answer using inverse trigonometric functions if necessary.)*

(a) $r(t) = \langle 1-t, t+2, -3t\rangle$, $s(u) = \langle u, 2u, u^2-u\rangle$

set $r(t) = s(u)$

$= \begin{cases} 1-t = u \rightarrow ① \\ t+2 = 2u \rightarrow ② \\ -3t = u^2 - u \end{cases}$

$1+2 = \quad 3 = 3u \rightarrow u = 1 \rightarrow$ plug to ①

$= \quad 1-t = 1$

$-t = 1-1 \rightarrow t = 0 \quad$ so $t=0, u=1$

$r(0) = \langle 1-0, 0+2, -3(0)\rangle = (1, 2, 0)$

$s(1) = \langle 1, 2(1), 1^2-1\rangle = (1, 2, 0)$

$\Big\}$ P.O.T $= (1, 2, 0)$

(b) Find angle $= \quad \vec{r}\cdot\vec{s} = |\vec{r}||\vec{s}|\cos\theta$

$\cos\theta = \dfrac{\vec{r}\cdot\vec{s}}{|\vec{r}||\vec{s}|}$

$r(t) = \langle 1-t, t+2, -3t\rangle$

$r'(t) = \langle -1, 1, -3\rangle$

$s(u) = \langle u, 2u, u^2-u\rangle$

$s'(u) = \langle 1, 2, 2u-1\rangle$

$s(1) = \langle 1, 2, 1\rangle$

$$\cos\theta = \frac{\langle -1, 1, -3\rangle \cdot \langle 1, 2, 1\rangle}{\sqrt{(-1)^2+(1)^2+(-3)^2}\,\sqrt{1^2+2^2+1^2}}$$

$$= \frac{-1+2-3}{\sqrt{11}\,\sqrt{6}} = -\frac{2}{\sqrt{66}} \rightarrow \cos\theta = -\frac{2}{\sqrt{66}}$$

$$\theta = \cos^{-1}\left(\frac{-2}{\sqrt{66}}\right)$$

$\downarrow$

for acute angle $\rightarrow \pi - \theta = \cos^{-1}\left(\frac{2}{\sqrt{66}}\right)$

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3. Let **r**(t) be a vector function that parameterizes a curve C. Give the definition of the unit tangent vector **T**(t) and prove that **T**(t) is always orthogonal to its derivative **T**$'(t)$.

As unit tangent vect $T(t) = \frac{r'(t)}{|r'(t)|}$

since $T(t)$ = unit vector

$$T(t)\cdot T(t) = 1$$

$$\frac{d}{dt} = T'(t)\cdot T(t) + T(t)\cdot T'(t) = 0$$

dot Product Commutative

$$= 2T'(t)\cdot T(t) = 0$$

\# Prove

Find an Arclength parametrization of the helix.

$$r(t) = \langle 2t, \cos(\pi t), \sin(\pi t)\rangle.$$

$$= \sqrt{\left(\frac{d}{dt}(2t)\right)^2 + \left(\frac{d}{dt}(\cos(\pi t))\right)^2 + \left(\frac{d}{dt}(\sin(\pi t))\right)^2}$$

$$= \sqrt{4 + (+\pi^2\sin^2(\pi t)) + (\pi^2\cos^2(\pi t))}$$

$$= \sqrt{4 + \pi^2(\underbrace{+\sin^2(\pi t) + \cos^2(\pi t)}_{1})}$$

$$s = \sqrt{4+\pi^2} \longrightarrow \text{Arclength} = \int_0^t \sqrt{4+\pi^2}\,dt$$

$$s = \sqrt{4+\pi^2}\,t \qquad t = \frac{s}{\sqrt{4+\pi^2}}$$

reparamatize $= \langle \frac{2s}{\sqrt{4+\pi^2}}, \cos\left(\frac{\pi s}{\sqrt{4+\pi^2}}\right), \sin\left(\frac{\pi s}{\sqrt{4+\pi^2}}\right) \rangle$.

Volume for Paravellpiped with side $i+j$, $4j-k$, $4i+3j-k$.

Solution

$= a \cdot (b \times c)$ later.

$a = \langle 1, 1, 0$

$=$

Distance from $3x+2y+5z-1=0$ to pnt $(2,0,-1)$

$$= \frac{3x+2y+5z-1}{\sqrt{9+4+25}} = \frac{3(2)+2(0)+5(-1)-1}{\sqrt{38}} = \frac{6-5-1}{\sqrt{38}} = \frac{0}{\sqrt{36}}$$

$(2,0,-1) \rightarrow$ is on the plane.

2. $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

Try to prove if a function all have the same path all limit D.N.E

when $x=0$ $\lim\limits_{y\to 0} = \frac{-y^2}{y^2} = -1$

when $y=0$ $\lim\limits_{x\to 0} = \frac{x^2}{x^2} = 1$

whe $y=x$ $\lim\limits_{x\to 0} \frac{x^2-x^2}{x^2+y^2} = 0$

as $-1 \neq 1 \neq 0$
$\rightarrow$ limit D.N.E

2(b) $\lim\limits_{(x,y,z,w)\to(0,0,0,0)} \frac{xyzw}{x^2+y^2+z^2+w^2}$

We try to show that the limit is equal to zero by bounding it to a function that tend to approach zero.

let set $c = x^2+y^2+z^2+w^2 \rightarrow c \to 0$

$\downarrow$

as $|x| = \sqrt{x^2} \to |x| \le \sqrt{x^2+y^2+z^2+w^2} = c^{\frac{1}{2}}$

Thus $\left| \dfrac{xyzw}{x^2+y^2+z^2+w^2} \right| \le \dfrac{c^{\frac{1}{2}} c^{\frac{1}{2}} c^{\frac{1}{2}} c^{\frac{1}{2}}}{c}$ (3)

$= c$

Thus $\left| \lim_{(x,y,z,w)\to(0,0,0,0)} \dfrac{xyzw}{x^2+y^2+z^2+w^2} \right| \le \lim_{(x,y,z,w)\to(0,0,0,0)} c = 0$ Thus

$$\lim_{(x,y,z,w)} \frac{xyzw}{x^2+y^2+z^2+w^2} = 0$$ #

Partial Derivative at the origin.

of $\boxed{f(x,y) = \dfrac{xy}{x^2+y^2}}$

$$\frac{\partial f}{\partial x}(0,0,0) = \underbrace{\lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}}_{\text{limit definition of partial derivative}} \longrightarrow \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,x) - f(x,x)}{h}$$

so apply this for $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ = limit approach to 0 #

In addition, For partial Derivative to be continuous

$$\lim_{\substack{(x,y)\to(0,0)\\ \downarrow \\ (a,b)}} f(x) = \frac{\partial f}{\partial x}(a,b) \quad \Big/ \quad \lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$ #

Def. "f has a continuous partial Derivatives" implies "f is differentiable" implies "f has partial derivative".

*** Quite tricky

(a) Let $f(x) = x^3$. Let $z = f(x^2 - y^2)$. use chain rule to compute $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$

$$z(\alpha) = f(\alpha) = \alpha^3 \quad , \quad \alpha(x,y) = x^2 - y^2$$

$$\begin{tikzcd} & & x \\ z \arrow[r] & \alpha \arrow[ur] \arrow[dr] & \\ & & y \end{tikzcd}$$

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial d} \cdot \frac{\partial d}{\partial y}$$

$$= 3\alpha^2 \cdot 2x + 3d^2 \cdot -2y$$

$$d = x^2 - y^2$$

$$= 3(x^2-y^2)^2 \cdot 2x + 3(x^2-y^2)^2 \cdot -2y$$

$$= 3(x^2-y^2)^2 (2x - 2y) \#$$

b.) Find the equation of the plane tangent to the surface defined by
$(x^2 + y^2 + 2z^2) = 4$ at $(1,1,1)$

find $\nabla_{f(x,y,z)}$ to get the vector of the tangent surface.

$$\nabla_{F(x,y,z)} = x^2 + y^2 + 2z^2 - 4 = 0$$

$$f_x = 2x \, , \quad f_y = 2y \, , \quad f_z = 4z$$

$$\nabla_f(x,y,z) = \langle 2x, 2y, 4z \rangle \quad \text{at } (1,1,1)$$

Gradient of the tangent plane toward the surface at point $(1,1,1)$

$$\downarrow$$

$$= \nabla_f(1,1,1) = \langle 2, 2, 4 \rangle$$

eq. of plane at pt (1, 1, 1)

tangent plane

$$= 2(x-1) + 2(y-1) + 4(z-1) = 0 \quad \#$$

$$= 2x - 2 + 2y - 2 + 4z - 4 = 0$$

$$= 2x + 2y + 4z - 8 = 0$$

$\#$

Find the limit Practice.

$$\lim_{(x,y)\to(1,0)} \frac{xy-y}{\sqrt{(x-1)^2+y^2}}$$

at $(x,y) = (1,0)$

$$\lim_{(x,y)\to(1,0)} = \frac{0-0}{\sqrt{0^2+0^2}} = \frac{0}{0} \text{ undefined}$$

so test different path to see for D.N.E.

at $x=0$ $\lim_{y\to 0} = \frac{0}{0}$ = undefined

at $y=0$ $\lim_{x\to 0} = \frac{0}{\sqrt{(x-1)^2}} = 0$ //

at $x=y$ $\lim_{y\to 0} = \frac{y^2-y}{\sqrt{(y-1)^2+y^2}} = \frac{y(y-1)}{\sqrt{(y-1)^2+y^2}} = \frac{y(y-1)}{\sqrt{y^2-2y+1+y^2}} = \frac{y(y-1)}{\sqrt{2y^2-2y+1}}$

$$\lim_{y\to 0} = \frac{0}{\sqrt{1}} = 0 \quad //$$

last path $y = kx$ $= \lim_{x\to 0} \frac{x(kx)-kx}{\sqrt{(x-1)^2+(kx)^2}} = \frac{kx^2-kx}{\sqrt{(x-1)^2+k^2x^2}} = \frac{kx(x-1)}{\sqrt{x^2-2x+1+k^2x^2}}$

$$= \frac{kx(x-1)}{\sqrt{x^2-2x+k^2x^2+1}}$$

$$\lim_{x\to 0} = \frac{0}{\sqrt{2}} = 0$$

$\#$

All limit thus approach toward zero $\#$

$$\lim_{(x,y)\to(1,0)} \frac{xy-y}{(x-1)^2+y^2}$$

Point (1,0) so +1 for the x-coordinate.

at $x=0$ $\lim_{y\to 0} = \frac{-y}{y^2} = \frac{-1}{y} =$ undefined

at $y=0$ $\lim_{x\to 0} = \frac{0}{(x-1)^2} = 0$

at $y=x$ $= \frac{x^2-x}{(x-1)^2+x^2}$

$= \frac{x(x-1)}{x^2-2x+1+x^2}$

$= \frac{x(x-1)}{2x^2-2x+1} = 0$

** Can also replace with $y=k(x-1)$ #

now test for $x = r\cos\theta + 1$

$y = r\sin\theta$

$= \frac{[(r\cos\theta+1)(r\sin\theta) - r\sin\theta]}{(r\cos\theta)^2+(r\sin\theta)^2}$

$= \frac{r^2\cos\theta\sin\theta}{r^2(\underbrace{\cos^2\theta+\sin^2\theta}_{1})}$

$= \cos\theta\sin\theta$

$\lim_{r\to 0} = \cos\theta\sin\theta$

as limit depend on the path θ, it doesn't approach certain value.

Thus, $0 \neq \theta$ so limit $\boxed{\text{D.N.E}}$

9 show that the limit D.N.E

$$\lim_{(x,y)\to(1,0)} \frac{(x-1)^2\log(x)}{(x-1)^2+y^2} \text{ if exist}$$

Squeeze theorem

as $\left|\frac{(x-1)^2}{(x-1)^2+y^2}\right| |\log(x)| \leq |\log(x)|$

$\lim_{(x,y)\to(1,0)} |\log(x)| = 0$

also $0 \leq \left|\frac{(x-1)^2\log(x)}{(x-1)^2+y^2}\right|$

Thus lim approach 0.

For $\lim_{(x,y)\to(0,0)} \dfrac{x^4y^3\sin x \cdot \cos y}{x^{10}+y^6}$

For path $x=0$

$$\lim_{y\to 0} = \frac{0}{y^6} = 0$$

path $y=0$

$$\lim_{x\to 0} = \frac{0}{x^{10}} = 0$$

For path $y=x$

$$\lim_{x\to 0} = \frac{x^4x^3\sin x \cdot \cos x}{x^{10}+x^6}$$

$$= \frac{\cancel{x^7}\sin x \cos x}{\cancel{x^7}(x^3+\frac{1}{x})} = \frac{\sin x \cos x}{x^3+\frac{1}{x}}$$

$$= \frac{\sin x \cos x}{\frac{x^4+1}{x}}$$

$$\lim_{x\to 0} = \frac{x(\sin x \cos x)}{x^4+1} = \frac{0}{1} = 0$$

For path $x = r\cos\theta$, $y = r\sin\theta$

$$\lim_{(x,y)\to(0,0)} = \frac{(r\cos\theta)^4(r\sin\theta)^3\sin(r\cos\theta)\cdot\cos(r\sin\theta))}{(r\cos\theta)^{10}+(r\sin\theta)^6}$$

$$= \frac{(r^4\cos^4\theta)(r^3\sin^3\theta)\, r\sin\theta\cos\theta \cdot r\cos\theta\sin\theta)}{r^{10}\cos^{10}\theta + r^6\sin^6\theta}$$

$$\downarrow$$

$$= 0$$

so limit overall migh approch to 0

Find If the limit exist or show D.N.E

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{4x^2+y^6}$$

let $x=0$

$$\lim_{y\to 0} = \frac{0}{y^6} = 0$$

check path 2 = $y = x^{\frac{2}{6}}$ $\lim_{x\to 0} = \frac{x(x^{\frac{2}{6}})^3}{4x^2+(x^{\frac{2}{6}})^6} = \frac{x^2}{4x^2+x^2} = \frac{\cancel{x^2}}{5\cancel{x^2}} \to \frac{1}{5}$

$0 \neq \frac{1}{5}$ prove that the limit different for each path

so lim → D.N.E

Midterm 2 Practice

1.) 2. (a) Find an equation of the surface consisting of all points in $\mathbb{R}^3$ that are equidistant from the point $(0,0,1)$ and the plane $z=2$.

> >| กลับไปที่หน้า 1

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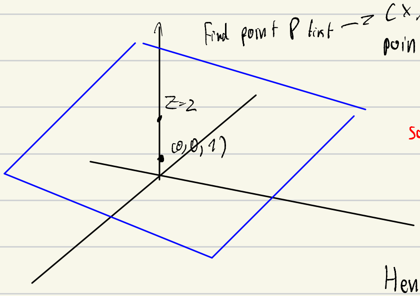
Find point P that $\rightarrow (x,y,z)$
point $(0,0,1)$



solution : The distance from a point $P=(x,y,z)$ to $(0,0,1)$

$$= \left.\sqrt{(x-0)^2+(y-0)^2+(z-1)^2}\right\} \text{need to be equidistant}$$

Distance to the plane $= z-2$

Hence eq =

$$\left(\sqrt{x^2+y^2+(z-1)^2}\right)^2 = (z-2)^2$$

$$x^2+y^2+(z-1)^2 = (z-2)^2$$

$$x^2+y^2+\cancel{z^2}-2z+1 = \cancel{z^2}-4z+4$$

$$x^2+y^2 = -2z+3$$

$$x^2+y^2-3 = -2z$$

$$\boxed{z = -\frac{x^2}{2}-\frac{y^2}{2}+\frac{3}{2}}$$

elliptic paraboloid, which direct toward the z axis.

3. Show that the function $\dfrac{x^{50}y^{50}}{x^{100}+y^{200}}$ does not have a limit at $(x,y)=(0,0)$.

test for path put $x=0$ $\lim_{y\to 0} = \dfrac{0}{y^{200}} = 0$

Path put $y=x$ $\lim_{x\to 0} \dfrac{x^{50}x^{50}}{x^{100}+x^{200}} = \dfrac{x^{100}}{x^{100}+x^{200}} = \dfrac{\cancel{x^{100}}}{\cancel{x^{100}}(1+x^{100})}$

$$= \frac{1}{1+x^{100}}$$

$$\lim_{x\to 0} = \frac{1}{1+0} = 1$$

as $\lim$ $0 \neq 1 \rightarrow$ Thus it doesn't have the same path

Hence : Prove D.N.E.

4. Consider the function $f(x,y) = x\cos(y) + y^2 e^x + x$.
(a) Find the differential of this function.

$$\left(df = \frac{\partial f}{\partial x} \cdot dx + \frac{\partial f}{\partial y} \cdot dy \right) \rightarrow$$ Differential Eq.

$$\frac{\partial f}{\partial x} = \cos(y) + y^2 e^x + 1 \quad , \quad \frac{\partial f}{\partial y} = -x\sin(y) + 2ye^x$$

$$df = (\cos(y) + y^2 e^x + 1)\,dx + (-x\sin(y) + 2ye^x)\,dy$$

b) as $\nabla f(x,y) = \langle \cos(y) + y^2 e^x + 1,\ -x\sin(y) + 2ye^x,\ 0 \rangle$

at point $(0, \pi, \pi^2)$

$$= \langle \cos(\pi) + \pi^2 e^0 + 1,\ -0\sin(\pi) + 2\pi e^0,\ 0 \rangle$$

$$= \langle \pi^2,\ 2\pi,\ 0 \rangle$$

so for tangent plane at $z := f(x,y) + f_x(x,y)(x-a) + f_y(x,y)(y-b)$

$$= f(0,\pi) + (\pi^2)(x-0) + 2\pi(y-\pi)$$

$f(0,\pi) = \pi^2$

$$= \pi^2 + \underbrace{(\pi^2)x + 2\pi(y-\pi)}$$

eq. for the tangent plane at $(0, \pi, \pi^2)$

$$= \pi^2 + \pi^2 x + 2\pi y - 2\pi^2$$

5. Suppose we need to know an equation of the tangent plane to a surface S at the point $P = (1, 3, 2)$. We don't have an equation for S, but we know that the curves

$$\mathbf{r}_1(t) = \langle 1 + 5t, 3 - t^2, 2 + t - t^3 \rangle,$$
$$\mathbf{r}_2(s) = \langle 3s - 2s^2, s + s^3 + s^4, s - s^2 + 2s^3 \rangle$$

both lie in S. Find an equation of the tangent plane to S at the point P.

find t that sastifiy each point for $r_1(t)$ and $r_2(t)$

for $r_1(t) \rightarrow$ $1+5t=1$, $3-t^2=3$, $2+t-t^3=2$

$t=0$ $t=0$ $t=0$

$r_2(s) \rightarrow$ $3s-2s^2=1$, $s+s^3+s^4=3$, $s-s^2+2s^3=2$

$s=1$ $s=1$ $s=1$

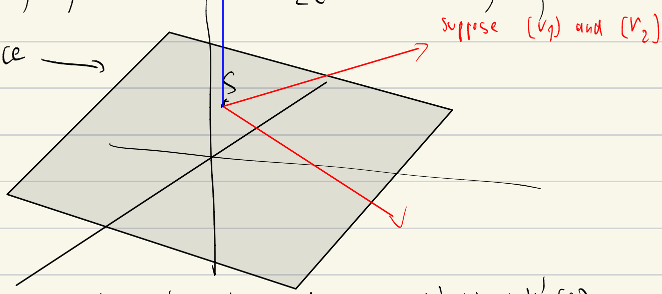
so sastify at $t=0$, $s=1$

find vector for r_1 and r_2

$r_1'(t) = \langle 5, -2t, 1-3t^2 \rangle$, $r_2'(s) = \langle 3-4s, 1+3s^2+4s^3, 1-2s+6s^2 \rangle$

$r_1(0) = \langle 5, 0, 3 \rangle$ $\vec{n}$ $r_2(1) = \langle -1, 8, 5 \rangle$

suppose surface $\longrightarrow$



to find normal vector to the plane $= r_1'(t) \times r_2'(s)$

$$= \langle 5, 0, 1 \rangle \times \langle -1, 8, 5 \rangle$$

$$\left|\begin{array}{ccc|cc} i & j & k & i & j \\ 5 & 0 & 1 & 5 & 0 \\ -1 & 8 & 5 & -1 & 8 \end{array}\right| =$$

Vector $= \langle -8, -26, 40 \rangle$

at point $P(1, 3, 2)$

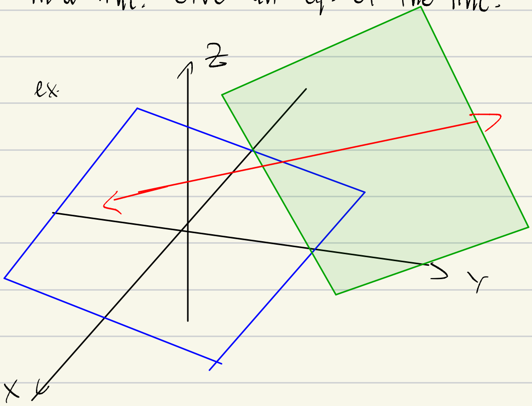
$\downarrow$

$= -8(x-1) - 26(y-3) + 40(z-2) = 0$

$\cancel{\#}$

7. The two planes, $2x - y + z = 0$ and $x + y + z = 6$ intersect in a line. Give an eq. of the line.

ex.



$2x - y + z = 0 \to ①$

$x + y + z = 6 \to ②$

let $z = 0$

① + ②

$= 3x + 2z = 6$

$3x = 6$

$x = 2$

normal vector of the 2 plane.

$\downarrow$

$= \langle 2, -1, 1 \rangle \times \langle 1, 1, 1 \rangle$

$= \left| \begin{matrix} i & j & k \\ 2 & -1 & 1 \\ 1 & 1 & 1 \end{matrix} \right| \begin{matrix} i & j \\ 2 & -1 \\ 1 & 1 \end{matrix} = -1i + 1j + 2k - (-1k + 1i + 2j)$

$= -2i - 1j + 3k$ at pt $(2, 4, 0)$

for $x + y + z = 6$

$2 + y + 0 = 6$

$y = 6 - 2 \to y = 4$

$x = 2, 4, 0$

$= x = 2 - 2t$

$y = 4 - t$

$z = 3t$

Example: Find the distance between the planes:

$P_1 : x-y+z=2 \Rightarrow \vec{n}_1 = \langle 1,-1,1 \rangle$ is a normal vector of P_1

$P_2 : 2x-2y+2z=3 \Rightarrow \vec{n}_2 = \langle 2,-2,2 \rangle = 2\vec{n}_1$ is a normal vector of P_2

$\vec{n}_1$ and $\vec{n}_2$ are colinear, hence P_1 and P_2 are parallel.

Let $P_1(x_1,y_1,z_1) \in P_1$. Hence $x_1-y_1+z_1=2$.

$$\text{dist}(P_1,P_2) = \text{dist}(P_1,P_2) = \left| \frac{2x_1-2y_1+2z_1+(-3)}{|\langle 2,-2,2 \rangle|} \right|$$

$$= \left| \frac{2(x_1-y_1+z_1)-3}{\sqrt{2^2+(-2)^2+2^2}} \right| = \left| \frac{4-3}{2\sqrt{3}} \right| = \frac{1}{2\sqrt{3}}.$$

Find the distance from the point $(1,-8,0)$ to the plane that is perpendicular to the curve $\vec{r}(t) = \langle t\cos t, \sin t, t+2 \rangle$ at $t=0$

$$\text{Distance} = \frac{|ax+by+cz+d|}{\sqrt{a^2+b^2+c^2}}$$

Find Eq. for the plane. Vector $= \vec{r}\,'(t) = \langle \cos t - t\sin t, \cos t, 1 \rangle$

at $t=0$

so $\vec{r}\,'(0) = \langle \cos(0) - 0\sin(0), \cos(0), 1 \rangle$

$= \langle 1, 1, 1 \rangle$

Thus the plane have a position vector at $\langle 1,1,1 \rangle$

$\vec{r}(0) = (0,0,2)$

Plane eq at point $\vec{r}(t) = x+y+z=2$

Point $(1,-8,0)$

$$\text{Dist} = \frac{|x+y+z-2|}{\sqrt{3}} = \frac{|1-8-2|}{\sqrt{3}} = \frac{|-9|}{\sqrt{3}}$$

$$= \frac{-9\sqrt{3}}{3} = +3\sqrt{3}$$

Midterm 1 for MATH 53

October 7, 2014

Show your work and justify your answers.

problem	points	score
1.	12	
2.	12	
3.	12	
4.	12	
5.	12	
6.	12	
7.	12	
8.	12	
XC	4	
total	96	

1. Find the area of the region enclosed by one loop of the curve $r^2 = \sin(2\theta)$.

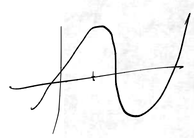
Set $r=0$ $\quad 0 = \sin(2\theta)$

$\theta = \frac{\pi}{2}$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2} r^2 \, d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2} \sin(2\theta) \, d\theta = \int_0^{\frac{\pi}{2}} \sin\theta \cos\theta \, d\theta$$

$u = \sin\theta \quad du = \cos\theta \, d\theta$

$$= \int_0^{\frac{\pi}{2}} u \, \cancel{\cos\theta} \, \frac{du}{\cancel{\cos\theta}}$$



$$= \frac{u^2}{2} \Big|_0^1 = \frac{1}{2}$$

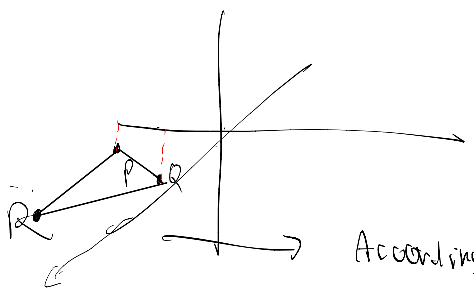
2. Decide if the triangle with vertices

$$P(0,-3,-4), Q(1,-5,-1), R(5,-6,-3)$$

is right-angled

(a) using angles between vectors

(b) using distances and the Pythagorean theorem.



$PQ = \langle 1, -2, 3 \rangle$ | $QR = \langle 4, -1, -2 \rangle$

$PR = \langle 5, -3, 1 \rangle$

According to the picture if $PQ \cdot QR = 0$ or $PQ \cdot PR = 0$ then it is orthogonal and $\theta = 90$

$PQ \cdot QR = 4 + 2 - 6 = 0$

Thus orthogonal & $\theta = 90$

3. Find an equation for the plane that passes through the point $(-2, 4, -3)$ and is perpendicular to the planes $-x + 3y - 5z = 42$ and $y - 2z = -5$.

plane 1 = $-x + 3y - 5z = 42$

Plane 2 = $y - 2z = -5$

Find Normal Vector that $\perp$ to plane 1 / plane 2

Plane 1 vector = $\langle -1, 3, -5 \rangle$, plane 2 vector = $\langle 0, 1, -2 \rangle$

$$N_1 \times N_2 = \begin{vmatrix} i & j & k \\ -1 & 3 & -5 \\ 0 & 1 & -2 \end{vmatrix} \begin{matrix} i & j \\ -1 & 3 \\ 0 & 1 \end{matrix} = -6\hat{i} + 0\hat{j} - \hat{k} - (0k - 5\hat{i} + 2\hat{j})$$

$$= -6\hat{i} + 0\hat{j} - \hat{k} - 0\hat{k} + 5\hat{i} - 2\hat{j}$$

$$= -\hat{i} - 2\hat{j} - \hat{k} \text{ or } \langle -1, -2, -1 \rangle$$

now Vector = $\langle -1, -2, -1 \rangle$ at pt pass through $(-2, 4, -3)$

$= eq.\ -1(x+2) - 2(y-4) - 1(z+3) = 0$ *

$-x - 2 - 2y + 8 - z - 3 = 0$

4. Let $\mathbf{r}(t) = \langle \sin t, 2\cos t \rangle$. $\quad -x - 2y - z + 3 = 0 \quad -x - 2y - z = -3$ *

(a) Sketch the plane curve with the given vector equation.

(c) Sketch the position vector $\mathbf{r}(t)$ and the tangent vector $\mathbf{r}'(t)$ for the value $t = \pi/4$ (use the same graph as for (a)) .

5. Find the limit, if it exists, or show that the limit does not exist.

(a) $\displaystyle \lim_{(x,y)\to(1,0)} \frac{xy - y}{(x-1)^2 + y^2}$

try $y = 0$

$\displaystyle \lim_{x \to 0} \frac{0}{(x-1)^2} = 0$ $\quad \neq \quad$ D.N.E

try $x = y + 1$

$\displaystyle \lim_{y \to 0} \frac{(y+1)y - y}{(y)^2 + y^2} = \frac{y^2 + y - y}{2y^2} = \frac{1}{2}$ //

(b)

$$\lim_{(x,y)\to(1,0)} \frac{xy-y}{\sqrt{(x-1)^2+y^2}}.$$

$$0 \leq \lim_{(x,y)\to(1,0)} \frac{xy-y}{\sqrt{(x-1)^2+y^2}}$$

as $|x-1| \leq \sqrt{(x-1)^2+y^2}$, $|y| \leq \sqrt{(x-1)^2+y^2}$

we can conclude that $\dfrac{|x-1|\,|y|}{\sqrt{(x-1)^2+y^2}} \leq \sqrt{(x-1)^2+y^2}$

By squeeze thm. $\lim_{(x,y)\to(1,0)} \sqrt{(x-1)^2+y^2} = 0 \to$ Thus limit approach 0.

6. Use the Chain Rule to find dw/dt. Express your answer solely in terms of the variable t.

$w = \ln\sqrt{x^2+y^2+z^2}$, $x = \sin t$, $y = \cos t$, $z = \tan t$.

$$\begin{tikzcd} & x \arrow[dr, dash] & \\ w \arrow[ur, dash] \arrow[r, dash] \arrow[dr, dash] & y \arrow[r, dash] & t \\ & z \arrow[ur, dash] & \end{tikzcd}$$

$$\frac{\sin t}{\cos t}$$

$$= \frac{\cos t\cos t + \sin t \sin t}{\cos^2 t}$$

$$= \frac{1}{\cos^2 t}$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$

$$\left[\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\left(\frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}}\cdot 2x\right)\right)\cdot\cos t\right] + \left[\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\left(\frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}}\cdot 2y\right)\right)\cdot -\sin t\right]$$

$$+\left[\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\left(\frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}}\cdot 2z\right)\right)\cdot \sec^2 t\right]$$

$$= \underbrace{\frac{x\cos t}{x^2+y^2+z^2} + \frac{-y\sin t}{x^2+y^2+z^2} + \frac{z\sec^2 t}{x^2+y^2+z^2}} = \frac{1}{\sec^2 t}\tan t\sec^2 t = \tan t.$$

อย่าลืม sub in x y z เข้า eq. ใหม่

7. Find the equations of (a) the tangent plane and (b) the normal line to the given surface at the specified point.

$$x^2 + y^2 + z^2 = 3xyz, \quad (1, 1, 1).$$

$$x^2 + y^2 + z^2 - 3xyz = 0$$

Find gradient $\nabla f(x, y, z)$

$\downarrow$

$f_x = 2x - 3yz$

$f_y = 2y - 3xz$

$f_z = 2z - 3xy$

$\nabla f(1,1,1) = \langle -1, -1, -1 \rangle$

plane eq. $= -1(x-1) - 1(y-1) - 1(z-1) = 0$

$-x + 1 - y + 1 - z + 1 = 0$

$-x - y - z = -3 \rightarrow (a)$ ∗

(b) Normal line

$x = 1 - t$

$y = 1 - t$

$z = 1 - t$

8. Find the extreme values of f on the region described by the inequality.

$$f(x, y) = 2x^2 + 3y^2 - 4x - 5, \quad x^2 + y^2 \leq 16.$$

9. (Extra Credit 4 pts.)

If $\mathbf{r}(t)$ is a 3-dimensional vector-valued function having all derivatives existing, and

$$\mathbf{u}(t) = \mathbf{r}(t) \cdot [\mathbf{r}'(t) \times \mathbf{r}''(t)],$$

show that

$$\mathbf{u}'(t) = \mathbf{r}(t) \cdot [\mathbf{r}'(t) \times \mathbf{r}'''(t)].$$

Math 53 – Practice Midterm 1 A – 80 minutes

Problem 1. (10 points)

Find the area enclosed by a loop of the curve given by the polar equation $r = \sqrt{\sin 2\theta}$.

Problem 2. (15 points)

a) Find the area of the space triangle with vertices $P_0 : (2, 1, 0)$, $P_1 : (1, 0, 1)$, $P_2 : (2, -1, 1)$.

b) Find the equation of the plane containing the three points P_0, P_1, P_2.

c) Find the intersection of this plane with the line parallel to the vector $\vec{V} = \langle 1, 1, 1\rangle$ and passing through the point $S : (-1, 0, 0)$.

Problem 3. (15 points)

a) Let $\vec{r} = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$ be the position vector of a path. Give a simple intrinsic formula for $\frac{d}{dt}(\vec{r}\cdot\vec{r})$ in vector notation (not using coordinates).

b) Show that if $\vec{r}$ has constant length, then $\vec{r}$ and $\vec{v}$ are perpendicular.

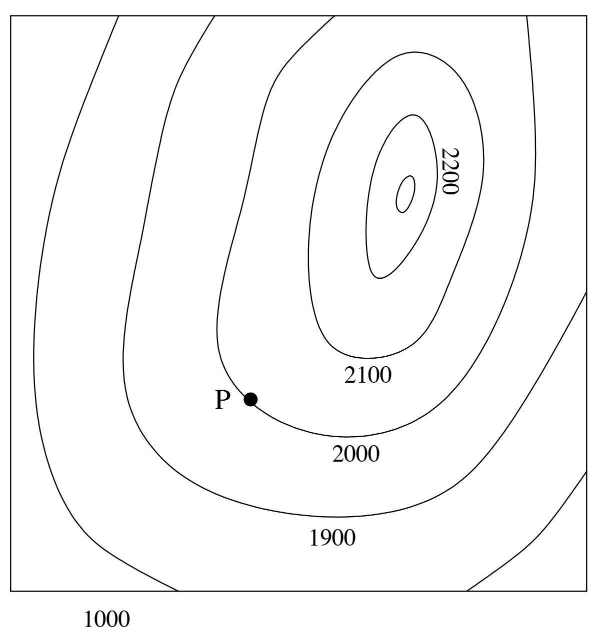
c) let $\vec{a}$ be the acceleration: still assuming that $\vec{r}$ has constant length, and using vector differentiation, express the quantity $\vec{r}\cdot\vec{a}$ in terms of the velocity vector only.

Problem 4. (10 points)

On the topographical map below, the level curves for the height function $h(x, y)$ are marked (in feet); adjacent level curves represent a difference of 100 feet in height. A scale is given.

a) Estimate to the nearest .1 the value at the point P of the directional derivative $D_{\hat{u}}h$, where $\hat{u}$ is the unit vector in the direction of $\hat{\imath} + \hat{\jmath}$.

b) Mark on the map a point Q at which $h = 2200$, $\frac{\partial h}{\partial x} = 0$ and $\frac{\partial h}{\partial y} < 0$. Estimate to the nearest .1 the value of $\frac{\partial h}{\partial y}$ at Q.



Problem 5. (10 points)

Let $f(x, y) = xy - x^4$.

a) Find the gradient of f at $P : (1, 1)$.

b) Give an approximate formula telling how small changes Δx and Δy produce a small change Δw in the value of $w = f(x, y)$ at the point $(x, y) = (1, 1)$.

Problem 6. (5 points)

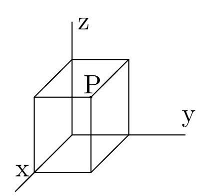
Find the equation of the tangent plane to the surface $x^3y + z^2 = 3$ at the point $(-1, 1, 2)$.

Problem 7. (5 points)

Let $w = f(u, v)$, where $u = xy$ and $v = x/y$. Express $\dfrac{\partial w}{\partial x}$ and $\dfrac{\partial w}{\partial y}$ in terms of x, y, f_u and f_v.

Problem 8. (20 points)

A rectangular box is placed in the first octant as shown, with one corner at the origin and the three adjacent faces in the coordinate planes. The opposite point $P : (x, y, z)$ is constrained to lie on the paraboloid $x^2 + y^2 + z = 1$. Which P gives the box of greatest volume?



a) Show that the problem leads one to maximize $f(x, y) = xy - x^3y - xy^3$, and write down the equations for the critical points of f. Critical Pnt = f_x and f_y

b) Find a critical point of f which lies in the first quadrant $(x > 0,\ y > 0)$, and determine its nature by using the second derivative test.

c) Find the maximum of f in the first quadrant (justify your answer).

Problem 9. (10 points)

In Problem 8 above, instead of substituting for z, one could also use Lagrange multipliers to maximize the volume $V = xyz$ with the same constraint $x^2 + y^2 + z = 1$.

a) Write down the Lagrange multiplier equations for this problem.

b) Solve the equations (still assuming $x > 0$, $y > 0$).

Problem 2. (15 points)

a) Find the area of the space triangle with vertices $P_0 : (2, 1, 0)$, $P_1 : (1, 0, 1)$, $P_2 : (2, -1, 1)$.

b) Find the equation of the plane containing the three points P_0, P_1, P_2.

c) Find the intersection of this plane with the line parallel to the vector $\vec{V} = \langle 1, 1, 1 \rangle$ and passing through the point $S : (-1, 0, 0)$.

a.) $\text{Area} = \dfrac{\| P_0P_1 \times P_0P_2 \|}{2}$

$P_0P_1 = \langle -1, -1, 1 \rangle$, $P_0P_2 = \langle 0, -2, 1 \rangle$

$P_0P_1 \times P_0P_2 = \begin{vmatrix} i & j & k \\ -1 & -1 & 1 \\ 0 & -2 & 1 \end{vmatrix} \begin{matrix} i & j \\ -1 & -1 \\ 0 & -2 \end{matrix} = -1\hat{i} + 0\hat{j} + 2\hat{k} - (0k - 2\hat{i} - 1\hat{j})$

$= 1\hat{i} + 1\hat{j} + 2\hat{k}$

$= \dfrac{\sqrt{1^2 + 1^2 + 2^2}}{2} = \dfrac{\sqrt{6}}{2}$

b.) plane eq. point: $(2, 1, 0)$, Vector: $\langle 1, 1, 2 \rangle$

$= 1(x-2) + 1(y-1) + 2(z) = 0$

c. Parametic eq for intersection line

as it is Parallel to $\vec{v}$ then that line also have vector $\langle 1, 1, 1 \rangle$ at $(-1, 0, 0)$

$$\left.\begin{aligned} x &= -1 + t \\ y &= t \\ z &= t \end{aligned}\right\} \text{line eq.}$$

sub into eq. for plane

$= 1(-1 + t - 2) + 1(t - 1) + 2(t) = 0$

$t - 3 + t - 1 + 2t = 0$

$4t = 4$

$t = 1$

$\text{POT} = (0, 1, 1)$

Problem 3. (15 points)

a) Let $\vec{r} = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$ be the position vector of a path. Give a simple intrinsic formula for $\frac{d}{dt}(\vec{r}\cdot\vec{r})$ in vector notation (not using coordinates).

b) Show that if $\vec{r}$ has constant length, then $\vec{r}$ and $\vec{v}$ are perpendicular.

c) let $\vec{a}$ be the acceleration: still assuming that $\vec{r}$ has constant length, and using vector differentiation, express the quantity $\vec{r}\cdot\vec{a}$ in terms of the velocity vector only.

Problem 4. (10 points)

a) $\frac{d}{dt}(\vec{r}\cdot\vec{r}) = \vec{v}\cdot\vec{r} + \vec{r}\cdot\vec{v}$

vector,

b) From def: $|\vec{r}| = C$ and continuous → This mean that

$r'(t)\cdot r(t) = 0$

Thus prove orthogonal

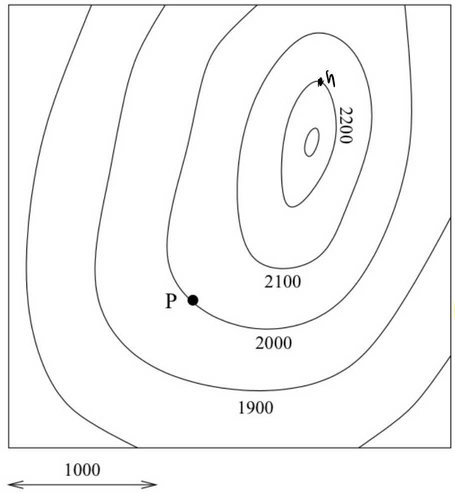
c) as $r\cdot v = 0$ → $\vec{r}\cdot\vec{a} = -|\vec{v}|^2$

Problem 4. (10 points)

On the topographical map below, the level curves for the height function $h(x, y)$ are marked (in feet); adjacent level curves represent a difference of 100 feet in height. A scale is given.

a) Estimate to the nearest .1 the value at the point P of the directional derivative $D_{\hat{u}}h$, where $\hat{u}$ is the unit vector in the direction of $\hat{\imath} + \hat{\jmath}$.

b) Mark on the map a point Q at which $h = 2200$, $\frac{\partial h}{\partial x} = 0$ and $\frac{\partial h}{\partial y} < 0$. Estimate to the nearest .1 the value of $\frac{\partial h}{\partial y}$ at Q.



a. $\Delta h = 100$

$\Delta s \approx 500,$

$D_u h \approx \frac{\Delta h}{\Delta s} \simeq 0.2$

b. $\frac{\partial h}{\partial y} \approx \frac{-100}{\frac{1000}{3}} \simeq -0.3$

Problem 5. (10 points)

Let $f(x, y) = xy - x^4$.

a) Find the gradient of f at $P : (1, 1)$.

b) Give an approximate formula telling how small changes Δx and Δy produce a small change Δw in the value of $w = f(x, y)$ at the point $(x, y) = (1, 1)$.

a. $\nabla_{f(x,y)}$ at $P: (1,1)$

$$f_x = y - 4x^3 \quad , \quad f_y = x$$

$$\nabla_f(1,1) = \langle -3, 1 \rangle$$

b. Formula linear approx: $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$$\text{Approx: } = -3(x-1) + 1(y-1)$$

$$\Downarrow$$

$$W = -3\Delta x + \Delta y$$

Problem 6. (5 points)

Find the equation of the tangent plane to the surface $x^3y + z^2 = 3$ at the point $(-1, 1, 2)$.

eq: $x^3y + z^2 - 3 = 0$ at pt. $(-1, 1, 2)$

$$= \nabla_{f(x,y,z)}$$

$$f_x = 3x^2y$$

$$f_y = x^3$$

$$f_z = 2z$$

so $\nabla_{f(x,y,z)} = \langle 3x^2y, x^3, 2z \rangle$ at point $(-1, 1, 2)$

$$\nabla_f(-1,1,2) = \langle 3(-1)^2(1), (-1)^3, 2(2) \rangle$$

$= \langle 3, -1, 4 \rangle$

eq to the plane at surface

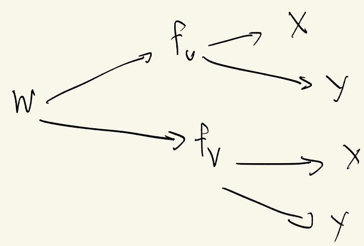
$= 3(x+1) - 1(y-1) + 4(z-2) = 0$

$= 3x + 3 - y + 1 + 4z - 8 = 0$

$3x - y + 4z = 4$

Problem 7. (5 points)

Let $w = f(u, v)$, where $u = xy$ and $v = x/y$. Express $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ in terms of x, y, f_u and f_v.



$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial f_u} \cdot \frac{\partial f(u)}{\partial x} + \frac{\partial w}{\partial f(v)} \cdot \frac{\partial f(v)}{\partial x}$$

$$= f_u \cdot y + f_v \frac{1}{y}$$

$$\frac{\partial w}{\partial y} = f_u u_y + f_v v_y = f_u x + -\frac{x}{y^2} f_v.$$

NAME: **STUDENT ID:**

MATH 53 1st MIDTERM

Please answer each question on a separate page – you can write on the back of the page. Remember to write your name and section number on EVERY page. Each problem is worth 10 points. Good Luck!

Problem 1. Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors in $\mathbb{R}^3$.

a) Show that $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = 0$. (You can use facts from the book/lectures as long as you *state them clearly.*)

We now from the properties of the cross product and vector addition that

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} + (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b}.$$

And we proved in class that

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0.$$

The conclusion follows.

b) if $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the standard basis vectors, find

$$\mathbf{i} \cdot (\mathbf{a} \times \mathbf{k}) + \mathbf{j} \cdot (\mathbf{a} \times \mathbf{i}) + \mathbf{k} \cdot (\mathbf{a} \times \mathbf{j})$$

in terms of a_1, a_2, a_3.

We recall the defintion (for $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$)

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle, \tag{1}$$

so that, since $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, $\mathbf{k} = \langle 0, 0, 1 \rangle$,

$$\mathbf{a} \times \mathbf{k} = \langle a_2, -a_1, 0 \rangle, \quad \mathbf{a} \times \mathbf{j} = \langle -a_3, 0, a_1 \rangle, \quad \mathbf{a} \times \mathbf{i} = \langle 0, a_3, -a_2 \rangle.$$

Hence,

$$\mathbf{i} \cdot (\mathbf{a} \times \mathbf{k}) = a_2, \quad \mathbf{j} \cdot (\mathbf{a} \times \mathbf{i}) = a_3, \quad \mathbf{k} \cdot (\mathbf{a} \times \mathbf{j}) = a_1,$$

and

$$\mathbf{i} \cdot (\mathbf{a} \times \mathbf{k}) + \mathbf{j} \cdot (\mathbf{a} \times \mathbf{i}) + \mathbf{k} \cdot (\mathbf{a} \times \mathbf{j}) = a_1 + a_2 + a_3.$$

c) Find the area of the *parallelogram* spanned by $\langle 1, 2, 3 \rangle$ and $\langle -1, 2, 3 \rangle$ (use the back page if needed).

The area of the parallelogram is given by the length of the cross product of the vectors defining the parallelogram. Hence, using

$$\begin{aligned} \text{Area} &= |\langle 1, 2, 3 \rangle \times \langle -1, 2, 3 \rangle| = |\langle 2 \cdot 3 - 3 \cdot 2, 3 \cdot (-1) - 1 \cdot 3, 1 \cdot 2 - 2 \cdot (-1) \rangle| \\ &= |\langle 0, -6, 4 \rangle| = \sqrt{0 + 36 + 16} = \sqrt{52} = \sqrt{4 \cdot 13} \\ &= 2\sqrt{13}. \end{aligned}$$

$$\begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ -1 & 2 & 3 \end{vmatrix} \begin{matrix} i & j \\ 1 & 2 \\ -1 & 2 \end{matrix} = 6i - 3j + 2k - (-2k + 6i + 3j)$$

$$= 6i - 3j + 2k + 2k - 6i - 3j$$

$$= \langle 0, -6, 4 \rangle$$

NAME: **STUDENT ID:**

Problem 2. a) Show that of the two curves parametrized by position vectors

$$\mathbf{r}_1(t) = \langle \cos(2\pi t), \sin(2\pi t), t\rangle, \qquad \mathbf{r}_2(t) = \langle t-1, t, t/4\rangle. \tag{2}$$

intersect at the point $(0, 1, 1/4)$ and find $\cos\theta$ where θ is the angle between the tangent vectors to the two curves at that point.

From the expressions for $\mathbf{r}_1$ and $\mathbf{r}_2$ we see that for the last component to be $\frac{1}{4}$ we have to take $t = \frac{1}{4}$ for $\mathbf{r}_1$ and $t = 1$ for $\mathbf{r}_2$. We then check that

$$\mathbf{r}_1(\tfrac{1}{4}) = \langle \cos(\pi/2), \sin(\pi/2), \tfrac{1}{4}\rangle = \langle 0, 1, \tfrac{1}{4}\rangle, \quad \mathbf{r}_2(1) = \langle 0, 1, \tfrac{1}{4}\rangle.$$

We compute tangent vectors by differentiating the position vectors:

$$\mathbf{r}_1'(t) = \langle -2\pi\sin(2\pi t), 2\pi\cos(2\pi t), 1\rangle, \quad \mathbf{r}_2'(t) = \langle 1, 1, \tfrac{1}{4}\rangle.$$

The tangent vectors at the point of intersection are given by

$$\mathbf{r}_1'(\tfrac{1}{4}) = \langle -2\pi, 0, 1\rangle, \quad \mathbf{r}_2'(1) = \langle 1, 1, \tfrac{1}{4}\rangle.$$

Hence

$$\cos\theta = \frac{\mathbf{r}_1'(\frac{1}{4})\cdot\mathbf{r}_2'(1)}{|\mathbf{r}_1'(\frac{1}{4})||\mathbf{r}_2'(1)|} = \frac{-2\pi + \frac{1}{4}}{(4\pi^2+1)^{\frac{1}{2}}(2+\frac{1}{16})^{\frac{1}{2}}} = \frac{-8\pi+1}{\sqrt{33}(4\pi^2+1)^{\frac{1}{2}}}.$$

b) Is there a value of t for which $\mathbf{r}_1'(t)$ and $\mathbf{r}_2'(t)$ are parallel? (Please use the back page if you need more space.)

For $\mathbf{r}_1'(t)$ and $\mathbf{r}_2'(t)$ to be parallel we need, for some scalar c,

$$\langle -2\pi\sin(2\pi t), 2\pi\cos(2\pi t), 1\rangle = c\langle 1, 1, \tfrac{1}{4}\rangle.$$

That means that $c = 4$ and

$$-2\pi\sin(2\pi t) = 2\pi\cos(2\pi t) = 4.$$

But $-\sin(2\pi t) = \cos(2\pi t)$ only if $2\pi t = 3\pi/4 + k\pi$. But then the value of $2\pi\cos(2\pi t) = \pm 2\pi/\sqrt{2} \neq 4$.

Problem 2. a) Show that of the two curves parametrized by position vectors

$$\mathbf{r}_1(t) = \langle \cos(2\pi t), \sin(2\pi t), t\rangle, \quad \mathbf{r}_2(t) = \langle t-1, t, t/4\rangle. \tag{2}$$

intersect at the point $(0, 1, 1/4)$ and find $\cos\theta$ where θ is the angle between the tangent vectors to the two curves at that point.

check if Intersect $\rightarrow$ $r_1(0,1,\frac{1}{4})$ should be equal to $r_2(0,1,\frac{1}{4})$

Find t for $r_1(t)$ and $r_2(t)$

for $r_1 \rightarrow t = \frac{1}{4}$ $\quad r_2 \rightarrow t = 1$

$r_1(\frac{1}{4}) = \langle 0, 1, \frac{1}{4} \rangle == r_2(1) = \langle 0, 1, \frac{1}{4} \rangle$

find Vector for $r_1(t)$ and $r_2(t)$

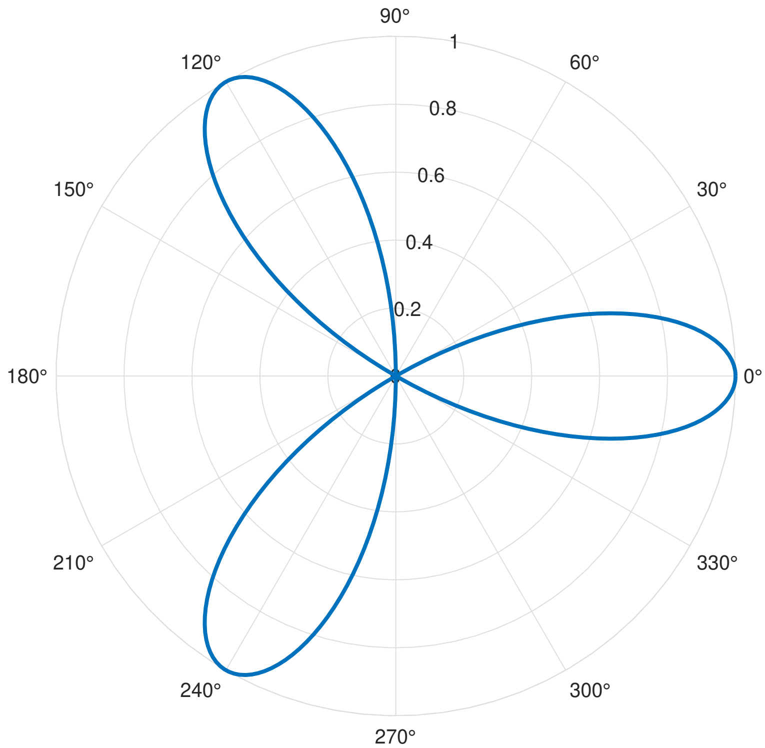
$= r_1'(t) = \langle -2\pi\sin(2\pi t), 2\pi\cos(2\pi t), 1 \rangle$, $r_2'(t) = \langle 1, 1, \frac{1}{4} \rangle$

$r_1'(\frac{1}{4}) = \langle -2\pi, 0, 1 \rangle \quad r_2' = \langle 1, 1, \frac{1}{4} \rangle$

$= \underbrace{\langle -2\pi, 0, 1 \rangle \cdot \langle 1, 1, \frac{1}{4} \rangle = \left(\sqrt{4\pi^2+1}\right)\left(\sqrt{2+\frac{1}{16}}\right)}_{\text{n solve MV}} \cos\theta$

NAME: **STUDENT ID:**

Problem 3. a) Sketch the polar curve given by the equation $r = \cos(3\theta)$, $0 \leq \theta \leq \pi$.



b) Compute the area enclosed by this curve. (Please use the other side of the page if needed.)

The area of the region defined by a polar curve $r = f(\theta)$, $\alpha \leq \theta \leq \beta$, is given by $\frac{1}{2}\int_\alpha^\beta f(\theta)^2 d\theta$. For $f(\theta) = \cos 3\theta$, $\alpha = 0$, $\beta = \pi$, gives

$$\text{Area} = \tfrac{1}{2}\int_0^\pi \cos^2(3\theta)d\theta = \tfrac{1}{6}\int_0^{3\pi} \cos^2 t dt.$$

Note that

$$\int_0^{3\pi} \cos^2 t dt = \int_0^{3\pi} \sin^2 t dt$$

and hence

$$\int_0^{3\pi} \cos^2 t dt = \tfrac{1}{2}\int_0^{3\pi} (\cos^2 t + \sin^2 t) dt = \tfrac{1}{2}\int_0^{3\pi} 1 dt = \tfrac{3}{2}\pi.$$

Hence

$$\text{Area} = \tfrac{1}{6}\tfrac{3}{2}\pi = \tfrac{1}{4}\pi.$$

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Problem 4. Match the following three dimensional curves to their equations. The parameter satisfies $0 \leq t \leq \pi$ for all curves.

a) $\mathbf{r}(t) = \langle t^3, t, t^3 \rangle$

b) $\mathbf{r}(t) = \langle t, t^3, t^3 \rangle$

c) $\mathbf{r}(t) = \langle \sin(t/2), t^3, t^3 \rangle$

d) $\mathbf{r}(t) = \langle t^3 \cos(t/2), t, t^3 \rangle$

Please do not guess: negative points will be given for wrong matches. We have three versions of the exam with different arrangements of answers!

Please use scratch paper and record your answers in the table below:

Figure 1	**Figure 2**	**Figure 3**	**Figure 4**
b	**c**	**a**	**d**

Figure 1

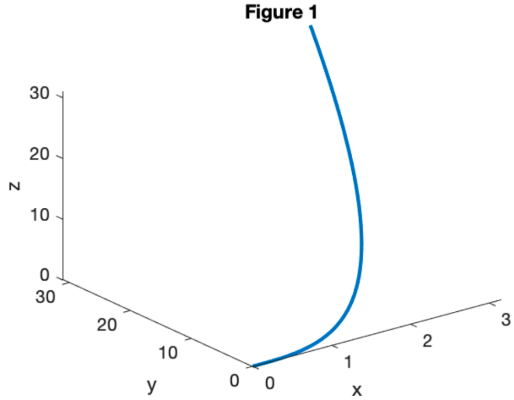


Figure 2

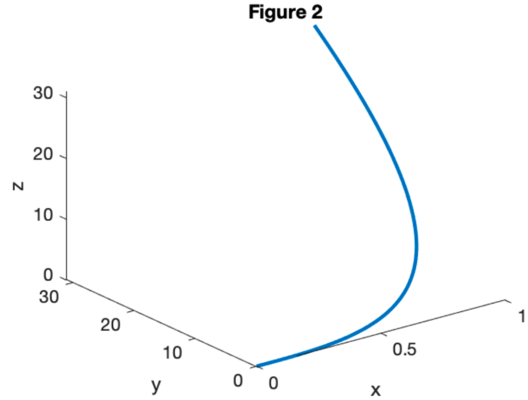


Figure 3

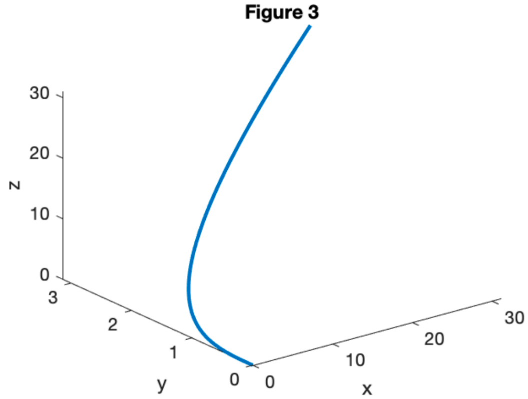
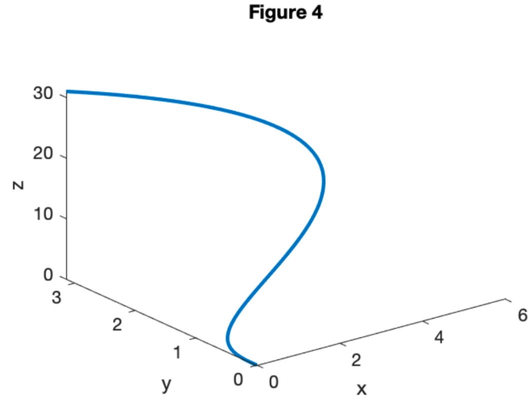


Figure 4



NAME: **STUDENT ID:**

Problem 5. a) Find the tangent plane to the graph of the function

$$f(x, y) = x^3 + 2x^2y + 2$$

at the point $(x_0, y_0, f(x_0, y_0))$ where $(x_0, y_0) = (1, -1)$.

The tangent plane to the graph of f at (x_0, y_0, z_0), $z_0 = f(x_0, y_0)$, is given by

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

In our case $x_0 = 1$, $y_0 = -1$ and $z_0 = 1^3 + 2 \cdot 1^2 \cdot (-1) + 2 = 1$. The partial derivatives are given by

$$f_x(x, y) = 3x^2 + 4xy, \quad f_y(x, y) = 2x^2,$$

so

$$f_x(1, -1) = -1, \quad f_y(1, -1) = 2.$$

The tangent plane is then

$$z - 1 = -(x - 1) + 2(y + 1) \text{ or } x - 2y + z - 4 = 0.$$

b) What is the distance of the point $(1, 2, 3)$ to the plane you found in part a).

The distance of a point (x_1, y_1, z_1) to the plane defined by $ax + by + cz + d = 0$ is given by

$$\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

In our case we found above that $a = 1$, $b = -2$, $c = 1$ and $d = -4$. Hence the distance of $(1, 2, 3)$ to the tangent plane is given by

$$\frac{|1 - 4 + 3 - 4|}{\sqrt{1 + 4 + 1}} = \frac{4}{\sqrt{6}}.$$

3. Find an equation of the plane containing the line

$$\frac{x-1}{2} = \frac{y+2}{3} = -z$$

and the point $(-2, 0, 5)$.

for line A vector = $\langle 2, 3, -1 \rangle$ and point = $(1, -2, 0)$

find $\overline{AB}$ = = $\langle 1, -2, 0 \rangle - (-2, 0, 5)$

Find a normal vector =

4. Find parametric equations of the line of intersection of the planes $3x - 2y + z = 1$ and $2x + y - 3z = 3$.

$$3x - 2y + z = 1$$
$$2x + y - 3z = 3$$

set $z = 0$

Thus
$$3x - 2y = 1$$
$$2(2x + y = 3)$$

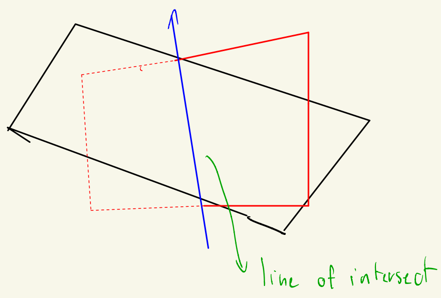
$$3x - 2y = 1$$
$$4x + 2y = 6$$
$$7x = 7$$
$$x = 1$$

$$3(1) - 2y = 1$$
$$-2y = 1 - 3$$
$$-2y = -2$$
$$y = 1$$

So $x = 1, y = 1, z = 0$ or Point of intersect $= (1, 1, 0)$

plane 1 vector
$\downarrow$
$\langle 3, -2, 1 \rangle$

Plane 2
$\downarrow$
$\langle 2, 1, -3 \rangle$



$$P_1 \times P_2 = \begin{vmatrix} i & j & k \\ 3 & -2 & 1 \\ 2 & 1 & -3 \end{vmatrix} \begin{matrix} i & j \\ 3 & -2 \\ 2 & 1 \end{matrix} = 6i + 2j + 3k - (-4k + i - 9j)$$
$$= 6\hat{i} + 2\hat{j} + 3\hat{k} + 4\hat{k} - \hat{i} + 9\hat{j}$$
$$= 5\hat{i} + 11\hat{j} + 7\hat{k} \quad \text{at Pnt } (1, 1, 0)$$

Parametric $=$
$$x = 1 + 5t$$
$$y = 1 + 11t$$
$$z = 7t$$

5. Find an equation for the surface consisting of all points P in the three-dimensional space such that the distance from P to the point $(0, -1, 0)$ is equal to the distance from P to the plane $y = 1$.

Identify this surface by name and sketch it.

Suppose $\{ P := (x, y, z) \in \mathbb{R}^3 \mid \theta : \mathbb{R} \}$

We can set up as Distance from p to $(0, -1, 0)$

$$= \sqrt{(x-0)^2 + (y+1)^2 + (z)^2}$$

$$= \sqrt{x^2 + (y+1)^2 + z^2}$$ → equidistant to $y = 1$

Thus $= \sqrt{x^2 + (y+1)^2 + z^2}^{\,2} = (y-1)^2$

$$x^2 + (y+1)^2 + z^2 = (y-1)^2$$

$$x^2 + z^2 + 4y = 0$$

Cone / Pallaboroid

6. Find the differential of the function $f(x, y, z) = \sqrt{x^2 + 4y^2 + z^2}$ and use it to approximate the number $f(1.98, 1.01, 1.02)$.

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

$$= \left(\frac{1}{2}(x^2+4y^2+z^2)^{-\frac{1}{2}} \cdot 2x\right) dx + \left(\frac{1}{2}(x^2+4y^2+z^2)^{-\frac{1}{2}} \cdot 8y\right) dy$$

$$+ \left(\frac{1}{2}(x^2+4y^2+z^2)^{-\frac{1}{2}} \cdot 2z\right) dz$$

$$df = \frac{x}{\sqrt{x^2+4y^2+z^2}} dx + \frac{4y}{\sqrt{x^2+4y^2+z^2}} dy + \frac{z}{\sqrt{x^2+4y^2+z^2}} dz$$

assume the point = $(2, 1, 1)$

$$df = \frac{2}{3} dx + \frac{4}{3} dy + \frac{1}{3} dz$$

Find change in dx, dy, dz

$-dx = 2 - 1.98$, $dy = 1.01 - 1 = 0.01$, $dz = 1.02 - 1 = 0.02$

$$df = \frac{2}{3}(-0.02) + \frac{4}{3}(0.01) + \frac{1}{3}(0.02) \approx 0.0067$$

approximation = $f(2,1,1) + df = 3.0067$

7. Write down an equation of the tangent plane to the surface $y = x^2z - 2xz^3 + z^2$ and the point $(2, 1, 1)$.

$$0 = x^2z - 2xz^3 + z^2 - y$$

find $\nabla f(x,y,z) \rightarrow$ $f_x = 2xz - 2z^3$

$$f_y = -1$$

$$f_z = x^2 - 6xz^2 + 2z$$

$\nabla f_{(2,1,1)} = \langle 2, -1, -6 \rangle$ at point $(2,1,1)$

$$= \quad 2(x-2) - 1(y-1) - 6(z-1) = 0$$

$$2x - 4 - y + 1 - 6z + 6 = 0$$

$$2x - y - 6z = -3$$

$$y = 2x - 6z + 3$$

$$-(y - 2x + 6z = 3)$$

$$2x - 6z - y$$

8. Let $f(x, y)$ be a function with continuous second partial derivatives. Suppose that $x = au + bv$ and $y = -bu + av$, where a and b are two real numbers such that $a^2 + b^2 = 1$. Show that

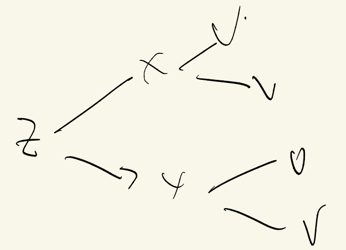
$$\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

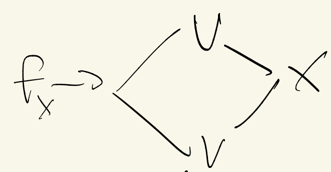
let $z = f(x,y) \rightarrow$ $x = au + bv$ and $y = -bu + av$.

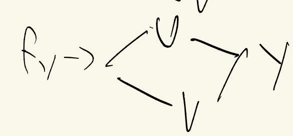
$$\frac{\partial}{\partial v}$$

$$\frac{\partial x}{\partial u} = a, \quad \frac{\partial x}{\partial v} = b, \quad \frac{\partial y}{\partial u} = -b, \quad \frac{\partial y}{\partial v} = a$$

as $a^2 + b^2 = 1$,







$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$= \frac{\partial f}{\partial x} \cdot a + \frac{\partial f}{\partial y} \cdot -b$$

$$\frac{\partial f}{\partial x} = \frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$= \frac{\partial f}{\partial x} \cdot b + \frac{\partial f}{\partial y} \cdot a$$

<u>Second Partial Derivative</u>: $\frac{\partial^2 f}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial u}\right)$, $\frac{\partial^2 f}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial v}\right)$

<u>For Partial f with respect to U</u> = $\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x} \cdot a + \frac{\partial f}{\partial y} - b\right)$

$$= a \cdot \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial x}\right) + \left(-b\left(\frac{\partial}{\partial u}\right)\left(\frac{\partial f}{\partial y}\right)\right)$$

δ

$$= a \cdot \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \cdot \underbrace{\frac{\partial x}{\partial U}}_{a} + -b \cdot \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \underbrace{\frac{\partial y}{\partial U}}_{-b}$$

$$= a^2 \frac{\partial^2 f}{\partial x^2} + b^2 \frac{\partial^2 f}{\partial y^2}$$

For Partial f with respect to V.

$$\frac{\partial}{\partial V}\left(\frac{\partial f}{\partial x} \cdot b + \frac{\partial f}{\partial y} \cdot a\right)$$

$$= b \cdot \frac{\partial}{\partial V}\left(\frac{\partial f}{\partial x}\right) + a \cdot \frac{\partial}{\partial V}\left(\frac{\partial f}{\partial y}\right)$$

$$= b \cdot \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \underbrace{\frac{\partial x}{\partial V}}_{b} + a \cdot \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \underbrace{\frac{\partial y}{\partial V}}_{a}$$

$$= b^2 \cdot \frac{\partial^2 f}{\partial x^2} + a^2 \frac{\partial^2 f}{\partial y^2}$$

respect to $U+V$ = $$a^2 \frac{\partial^2 f}{\partial x^2} + b^2 \frac{\partial^2 f}{\partial y^2} + b^2 \frac{\partial^2 f}{\partial x^2} + a^2 \frac{\partial^2 f}{\partial y^2}$$

$$= (a^2+b^2) \frac{\partial^2 f}{\partial x^2} + (a^2+b^2) \frac{\partial^2 f}{\partial y^2}$$

as $a^2+b^2=1$

Thus Prove $= \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2}$ //