Exam 3 : Practice test 1.

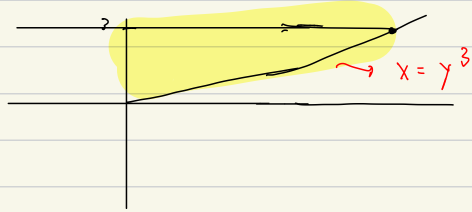
Reverse the order of integration and then evaluate the integral. Be sure to include a 2D graph of the integration region.

1) $\displaystyle\int_0^{27}\int_{\sqrt[3]{x}}^{3} \frac{1}{y^4+1}\,dy\,dx$

$0 \le x \le 27$

$x^{\frac{1}{3}} \le y \le 3$

as $(y)^3 = \left(x^{\frac{1}{3}}\right)^3 \Rightarrow x = y^3$



$$\underline{\text{Type 2}} = \int_0^3\int_0^{y^3} \frac{1}{y^4+1}\,dx\,dy$$

$$= \int_0^3 \frac{y^3}{(y^4+1)}\,dy \longrightarrow \frac{1}{4}\int_0^3 \frac{1}{u}\,du$$

$$\downarrow$$

$u = y^4+1 \qquad du = 4y^3\,dy$

$dy = \dfrac{du}{4y^3}$

$$= \frac{1}{4}\left[\ln(u)\Big|_1^{82}\right]$$

$$= \frac{1}{4}\left[\ln(82)\right]$$

$$= \frac{\ln(82)}{4}$$

\#

Change the Cartesian integral to an equivalent polar integral, and then evaluate.

2)

$$\int_0^{\ln 7}\int_0^{\sqrt{(\ln 7)^2 - y^2}} e^{\sqrt{x^2+y^2}}\,dx\,dy$$

Be sure to include a 2D graph of the integration region.

cheatsheet

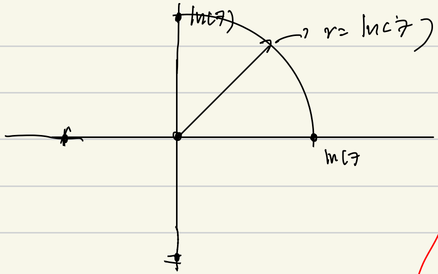
$$x^2 = \sqrt{(\ln 7)^2 - y^2}^{\,2}$$

$$x^2 = (\ln 7)^2 - y^2$$

$$x^2 + y^2 = (\ln 7)^2$$

$$\sqrt{r^2} = \sqrt{(\ln 7)^2}$$

$$r = \ln(7) \longrightarrow \quad 0 \le r \le \ln(7)$$



Note

eq: like this

$\downarrow$

$\sqrt{(\ln 7)^2 - y^2}$

$\downarrow$

right half of the circle with radius of ln(7)

$y = \pm\sqrt{r^2 - x^2}$

$+$ ∩ $\quad -$ ∪

$x = \pm\sqrt{r^2 - y^2}$

$+$) $\quad -$ (

$$= \int_0^{\frac{\pi}{2}}\int_0^{\ln(7)} e^r\, r\, dr\, d\theta$$

$$= \frac{\pi}{2}\int_0^{\ln(7)} r e^r\, dr$$

$\downarrow$

$u = r \quad dv = e^r$

$du = dr \quad v = e^r$

Note: $\ln(e^x) = x$

$e^{\ln(x)} = x$

$$r e^r - e^r$$

$$= \frac{\pi}{2}\left[(re^r - e^r)\Big|_0^{\ln(7)}\right] = \frac{\pi}{2}\left[\left[\ln(7)\,\cancel{e^{\ln(7)}}^{\,7} - \cancel{e^{\ln(7)}}^{\,7}\right] - [-1]\right]$$

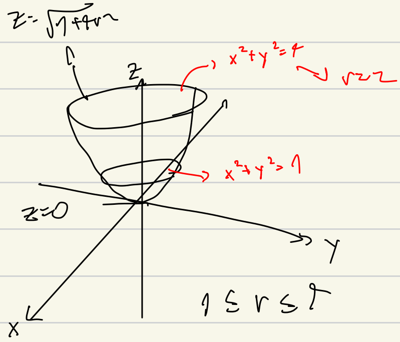
$$= \frac{\pi}{2}\left[7\ln(7) - 6\right]$$

#

Solve the problem.

3) Find the surface area of the part of the hyperbolic paraboloid $z = y^2 - x^2$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. You must also include a 2D graph of the integration region.

Surface area $= \sqrt{1 + (f_x)^2 + (f_y)^2}$



$1 \le r \le 2$

$z = y^2 - x^2$

$f_x = -2x\ dx$

$f_y = 2y\ dy$

Surface $= \sqrt{1 + (-2x)^2 + (2y)^2}$

$= \sqrt{1 + 4x^2 + 4y^2}$

$= \sqrt{1 + 4r^2}$

$$= \int_0^{2\pi} \int_1^2 \sqrt{1+4r^2}\ r\ dr\ d\theta$$

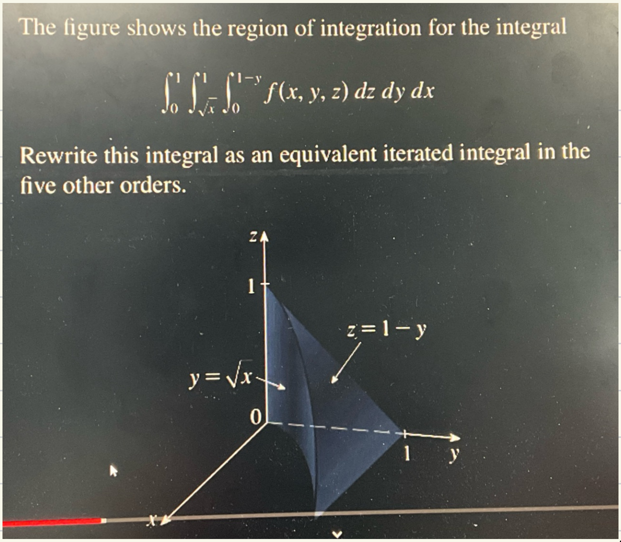
$$= 2\pi \left[\int_1^2 r\sqrt{1+4r^2}\ dr \right]$$

$u = 1+4r^2 \quad du = 8r\ dr$

$dr = \dfrac{du}{8r}$

$$= \frac{\pi}{4} \left[\int_1^2 u^{\frac{1}{2}}\ du \right]$$

$$= \frac{\pi}{\cancel{4}_{2}} \left[\frac{\cancel{2}\, u^{\frac{3}{2}}}{\cancel{3}} \Bigg|_5^{17} \right] = \frac{\pi}{6} \left[17^{\frac{3}{2}} - 5^{\frac{3}{2}} \right]_{//}$$



$z = 1 - y$

$y = 1 - z$

$0 \le z \le 1 - y$

$\sqrt{x} \le y \le 1 \to y = \sqrt{x}$

$0 \le x \le 1$

$\sqrt{x} = 1 - z$

$z = 1 - \sqrt{x}$

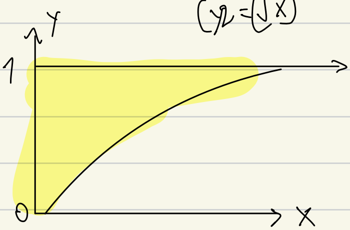
$(\sqrt{x})^2 = (1 - z)^2$

$x = (1 - z)^2$

$(y)^2 = (\sqrt{x})^2 \quad x = y^2$

Start with type 1. (x, y)

$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x, y, z)\, dz\, dy\, dx$

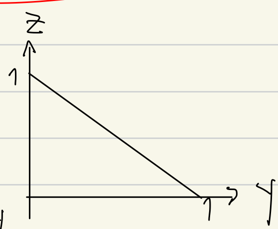


Type 2 $\quad 0 \le x \le y^2$

1. ※ $\int_0^1 \int_0^{y^2} \int_0^{1-y} f(x, y, z)\, dz\, dx\, dy$

Note that in triple integral, the outside part always constant.

Type 2. $\quad y - z$



Type 1 $\quad dz\, dy$

$0 \le z \le 1 - y$

$0 \le y \le 1$

2※ $= \int_0^1 \int_0^{1-y} \int_0^{y^2} f(x, y, z)\, dx\, dz\, dy$

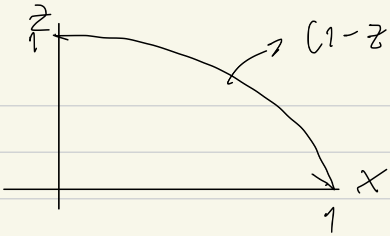
Type 2 $\quad dy\, dz$

$0 \le y \le 1 - z$

$0 \le z \le 1$

※3 $\int_0^1 \int_0^{1-z} \int_0^{y^2} f(x, y, z)\, dx\, dy\, dz$

Type 3 xz



$(1-z)^2 \longrightarrow \sqrt{x} = \sqrt{(1-z)^2}$

$\sqrt{x} = 1 - z$

$\downarrow$

$z = 1 - \sqrt{x}$

Type 1 dx-dz

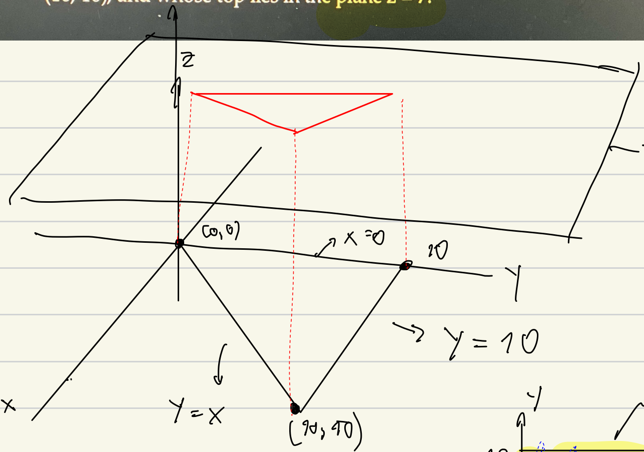
$0 \le x \le (1-z)^2$

$$\#4 = \int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{(1-z)} f(x,y,z)\, dy\, dx\text{-}dz$$

$$\#5 = \int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f(x,y,z)\, dy\, dz\, dx$$

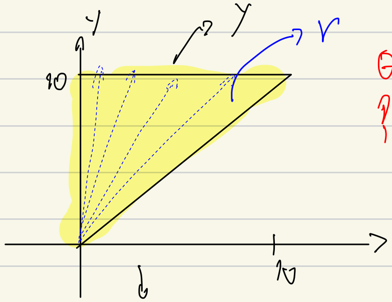
Set up ONLY the iterated integral for evaluating $\iiint f(r, \theta, z)\, dz\, r\, dr\, d\theta$ over the given region E. You must include a 2D graph of the base of the region (3D graph optional).

5) E is the rectangular solid whose base is the triangle with vertices at (0, 0), (0, 10), and (10, 10), and whose top lies in the plane z = 7.



$\to z = 7$

เช็คจากดูก (x-y) ก่อน //



θ = จากเริ่มถึง ตรง จุดขอบ

$$\int_{\pi/4}^{\pi/2} \int_{0}^{10\csc\theta} \int_{0}^{7} f\, r\, dz\, dr\, d\theta$$

$\theta = \frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

$0 \le r \le y = 10$

Polar Coordinate

$r\sin\theta = 10$

$r = \frac{10}{\sin\theta} = 10\csc\theta$

Use a spherical coordinate integral to find the volume of the given solid. You must include a 3D labeled graph of the solid.

6) the solid bounded below by the sphere $\varrho = 2\cos\varphi$ and above by the cone $z = \sqrt{x^2 + y^2}$

Note

Cone $\Rightarrow \frac{\pi}{4}$

$\rho = 2\cos\phi$

Center at z axis

$(0, 0, 1)$

radius $= 1$

$\rho^2 = 2\rho\cos\phi$

$x^2 + y^2 + z^2 = 2z$

$x^2 + y^2 + z^2 - 2z + 1 = 1$

$(x^2 + y^2 + (z-1)^2 = 1$

$z^2 = x^2 + y^2$

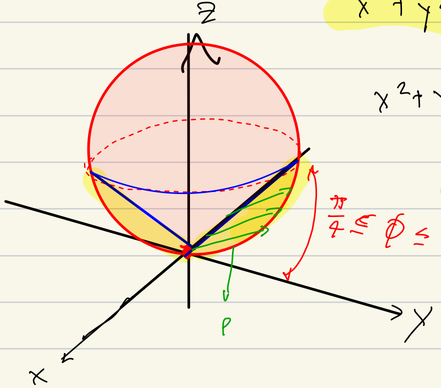
$z^2 + z^2 = 2z$

$\cancel{2}z^2 = \cancel{2}z$

$z^2 - z = 0$

$z(z-1) = 0$

$z = 1$



$\rho: 0 \le \rho \le 2\cos\phi$

$$= \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 2\pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi$$

$$= \frac{2\pi}{3} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2\cos\phi)^3 \sin\phi \, d\phi$$

$$= \frac{16\pi}{3} \int_{\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^3\phi \sin\phi \, d\phi$$

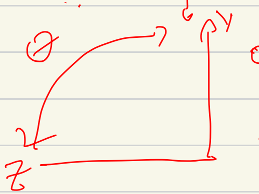
$u = \cos\phi$

$$\longrightarrow -\frac{\cancel{16}\pi}{3} \cdot \frac{1}{\cancel{4}}\left[\cos^4\phi\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= -\frac{4\pi}{3}\left[0 - \left(\frac{1}{\sqrt{2}}\right)^4\right]$$

Note $\phi = \frac{\pi}{4}$ is a cone

To find ϕ we start from positive z axis to y.



$0 \le \theta \le \frac{\pi}{2}$

$$= \frac{4\pi}{3}\left(\frac{1}{4}\right) = \frac{\pi}{3}$$

Use the given transformation to evaluate the integral. No graphs required.

7) $x = 2u$, $y = 5v$, $z = 4w$;

$$\iiint_R \left(\frac{x^2}{4} + \frac{y^2}{25} + \frac{z^2}{16}\right)^{\pi} dx\, dy\, dz,$$

where R is the interior of the ellipsoid $\frac{x^2}{4} + \frac{y^2}{25} + \frac{z^2}{16} = 1$

Jacobian $\Rightarrow$ $x = 2u$, $y = 5v$, $z = 4w$;

$$\frac{\cancel{(2u)^2}}{\cancel{4}} + \frac{\cancel{(5v)^2}}{\cancel{25}} + \frac{\cancel{(4w)^2}}{\cancel{16}} = u^2 + v^2 + w^2 = 1$$

$$xyz \to uvw \longrightarrow \rho, \theta, \phi$$

$$\begin{vmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{vmatrix} = 2(5(4) - 0,0) = 40$$

$$= \iiint_S (u^2 + v^2 + w^2)^{\pi} \, |40| \cdot dV$$

$$\downarrow$$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{1} (\rho^2)^{\pi} \cdot 40 \cdot \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$$

$$= 80\pi \left[\int_0^{\pi} \sin\phi\, d\phi\right]\left[\int_0^{1} \rho^{2\pi + 2}\, d\rho\right)$$

$$\downarrow$$

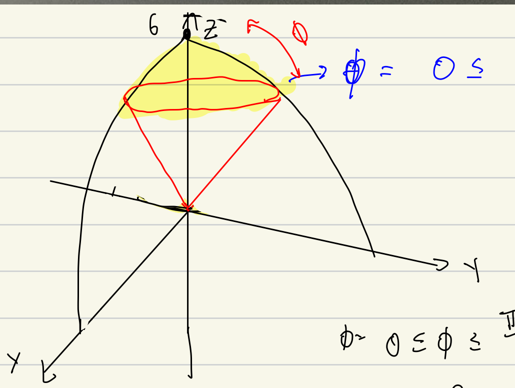
$$= \frac{80\pi}{2\pi+3}\Big[\cos\phi\Big]_{\pi}^{0}\Big[\rho^{2\pi+3}\Big]_{0}^{1}$$

$$(1-(-1)) \qquad (1-0)$$

$$\text{Ans} = \frac{160\pi}{2\pi+3}$$

8) Let D be the region that is bounded below by the cone $\varphi = \frac{\pi}{4}$ and above by the sphere $\varrho = 6$. Set up the triple integral for the volume of D in spherical coordinates. You must include a labeled sketch of the solid.

$\rho = 6$



$\phi = 0 \le$

$\phi: 0 \le \phi \le \frac{\pi}{4}$

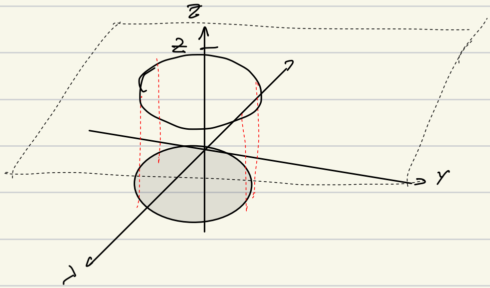
$\theta: 0 \le \theta \le 2\pi$

$\rho: 0 \le \rho \le 6$

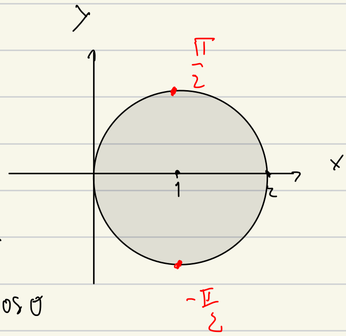
$$\boxed{\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{6} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}$$

Set up a triple integral to find the volume of the indicated region. You must include a label sketch of the integration region (2D and 3D). Do not evaluate the integral.

9) the region bounded below by the xy–plane, laterally by the cylinder $r = 2\cos\theta$, and above by the plane $z = 7$



$0 \le r \le 2\cos\theta$



$\theta = -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$r = 0 \le r \le 2\cos\theta$

$z = 0 \to 7$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{2\cos\theta} \int_{0}^{7} r \, dz \, dr \, d\theta$$

10) Evaluate

$$\int_{-7}^{7}\int_{-\sqrt{49-x^2}}^{\sqrt{49-x^2}}\int_{\sqrt{x^2+y^2}}^{7} dz\,dy\,dx$$

by transforming to cylindrical or spherical coordinates. You must include a labeled graph of the integration region.

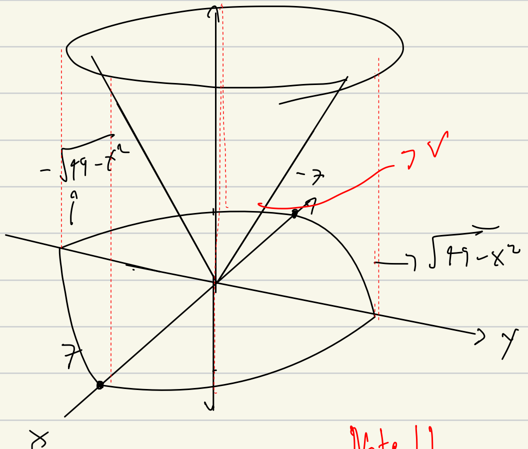
$y = \sqrt{49 - x^2}$

$y^2 = 49 - x^2$

$x^2 + y^2 = 49$

$r^2 = 49$

$r = 7 \longrightarrow \quad 0 \le r \le 7$



Note !!

Con eq. $= \sqrt{y^2 + x^2}$

$z = \quad r \le z \le 7$

$0 \le r \le 7$

$0 \le \theta \le 2\pi$

$$\int_0^{2\pi}\int_0^{7}\int_r^{7} r\,dz\,dr\,d\theta$$

$$= 2\pi \int_0^7 7r - r^2\,dr$$

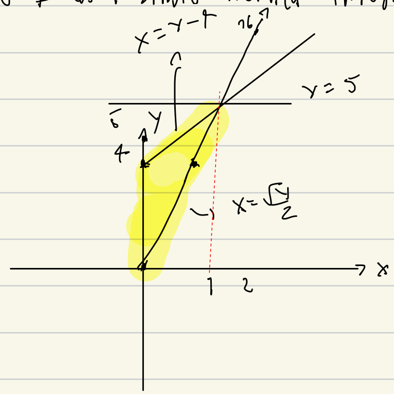
$$= 2\pi \left[\frac{7}{2}r^2 - \frac{1}{3}r^3\right]_0^7 \Rightarrow = \frac{343\pi}{3}$$

Exam 2 ; Practice (Multiple integral)

$$I = \int_0^4 \int_0^{\frac{\sqrt{y}}{2}} \frac{5}{4+x-4x^2}\,dx\,dy + \int_4^5 \int_{y-4}^1 \frac{5}{4+x-4x^2}\,dx\,dy.$$

(a) (4 points) Identifying equations of all the boundaries and the coordinates of their points of intersection, sketch the region of integration in the xy-plane.

(b.) Write I as a single iterated integral and then evaluate it.



$I =$

$x = \frac{\sqrt{y}}{2}$

$(2x)^2 = \sqrt{y}^2$

$4x^2 = y$

$x = y - 4$

$y = x + 4$

(b.)

$$= \int_0^1 \int_{4x^2}^{x+4} \frac{5}{4+x-4x^2}\,dy\,dx$$

$\frac{\sqrt{y}}{2} = x$

$2x = \sqrt{y}$

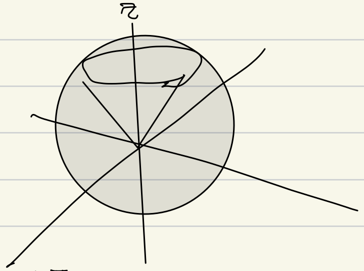
$4x^2 = y$

$$= 5\left[\int_0^1 \left[\frac{x+4}{4+x-4x^2} - \frac{4x^2}{4+x-4x^2}\right] dx\right]$$

$$= 5\left[\int_0^1 \frac{-4x^2 \cancel{+x} + 4}{4 \cancel{+x} - 4x^2}\, 1\,dx\right] = 5x$$

2A. Let D be the region that is bounded below by the cone $\phi = \frac{\pi}{4}$ and above by the sphere $\rho = 3$. Set up but do not solve a triple integral to find the volume of D:

(a) (5 points) In spherical coordinates.

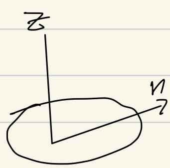


a.) $$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{3} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta \quad \#$$

b.) In cylindrical coordinate.

$\rho^2 = x^2+y^2+z^2$

$\rho = \sqrt{x^2+y^2+z^2}$



$$\rho^2 = 3\rho$$

$$(x^2+y^2+z^2)^2 = (3\sqrt{x^2+y^2+z^2})^2$$

$$\frac{(x^2+y^2+z^2)^2}{\cancel{(x^2+y^2+z^2)}} = 9$$

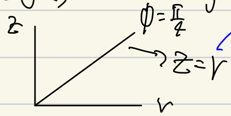
$$x^2+y^2+z^2 = 9$$

$$z^2 = 9-x^2-y^2$$

$$z = \sqrt{9-x^2-y^2} \rightarrow z: \; 0 \le z \le \sqrt{9-x^2-y^2}$$

as $z = \sqrt{9-x^2-y^2}$

when $z = 0 \rightarrow$



$z = \sqrt{9-r^2}$

as $z = r$

$r^2 = \sqrt{9-r^2}$

$r^2 + r^2 = 9$

$2r^2 = 9 \rightarrow r = \frac{3}{\sqrt{2}}$

$r: \; 0 \le r \le \frac{3}{\sqrt{2}}$

$$= \int_0^{2\pi}\int_0^{\frac{3}{\sqrt{2}}}\int_0^{\sqrt{9-r^2}} r \, dz\, dr\, d\theta \quad \#$$

Green theorem

2nd Partial derivative

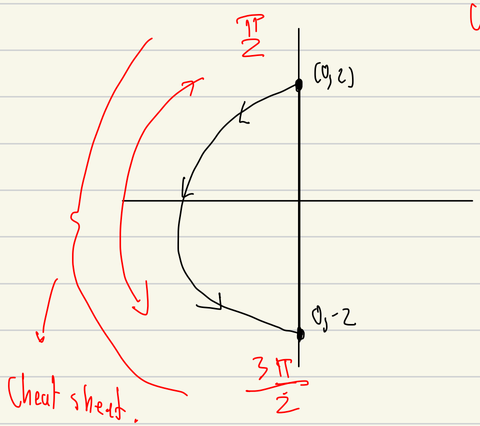
3A. Let C be the closed curve that starts at $(0,2)$, goes along the left half of a circle of radius 2 centered at the origin to $(0,-2)$, and returns to $(0,2)$ along the y-axis. Find

$$\oint_C (\arctan(x) - x^2y)\,dx + (y^2x + 1 + e^{y-\cos y})\,dy.$$

Check orientation => Positive.

As it is a simple close curve, positive orientation counter clockwise and piece wise smooth. Hence we can apply the green theorem

Closed curve = start and stop at same point



$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

$$\int_C P\,dx + Q\,dy = \iint_R Q_{dx} - P_{dy}\,dA$$

$$= \iint_R y^2 - (-x^2)$$

$$= \iint_R y^2 + x^2$$

$$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^2 r^2 r\,dr$$

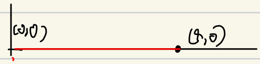
$$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 4\,d\theta = 4\left(\frac{3\pi}{2}\right) - 4\left(\frac{\pi}{2}\right)$$

$$= 6\pi - 2\pi = 4\pi$$

: Green theorem => Double Integral.

4A. Let $\overset{P}{M}(x,y) = x + 2xy\cos(x^4y^2)$ and $\overset{Q}{N}(x,y) = y^3 + x^2\cos(x^4y^2)$.

(a) (3 points) Find $\int_{C_1} M\,dx + N\,dy$ if C_1 is the straight line from $(4,0)$ to $(0,0)$.



※ Segment $\quad (1-t)(4,0) + t(0,0)$

$= \langle 4-4t, 0\rangle + \langle 0,0\rangle \quad t \in [0,1)$

$= \langle 4-4t, 0\rangle$

$dx = -4, \quad dy = 0$

a. $\underline{\int_{C_1} P\,dx + Q\,dy}$

$$= \int_0^1 (4-4t)(-4)\,dt$$

$$= \int_0^1 -16 + 16t \,dt$$

$$= -16t + 8t^2 \Big|_0^1 = -16 + 8 = -8 \quad \#$$

(b) (3 points) Find $\underline{\int_{C_2} M\,dx + N\,dy}$ if C_2 is the straight line from $(0,0)$ to $(0,2)$.

$\underline{\text{Segment.}}$ $\quad (1-t)(0,0) + t(0,2)$

$= (0,0) + (0,2t)$

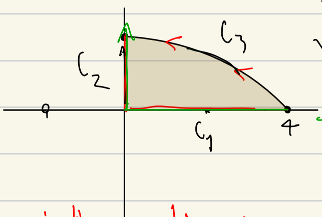
$= \langle 0, 2t\rangle$

$dx = 0 \quad dy = 2 \quad t \in [0,1)$

$$= \int P\,dx + Q\,dy = \int (2t)^3 \cdot 2 = \int_0^1 8t^3 \cdot 2 = \int_0^1 16t^3$$

$$= 4t^4 \Big|_0^1 = 4 \quad \#$$

(c) (4 points) Find $\int_{C_3} M\,dx + N\,dy$ if C_3 is the part of the parabola $x = 4 - y^2$ going from $(4,0)$ to $(0,2)$. You may use without verifying the fact that $M_y = N_x$.



$y^2 = 4 - x$

$y = \sqrt{4-x}$

line integral are curve independent.
so we can choose our own line that start from same p.n.t and end at same p.n.t but just different directions.

Hence: in this questions we use the combination of C_1 and C_2.

$$\int_{C_3} M\,dx + N\,dy = \int_{C_1} M\,dx + N\,dy + \int_{C_2} M\,dx + N\,dy$$

$$-8 + 4 = -4 \ \#$$

5A. Consider the double integral

$$I = \iint_R (x-y)(x+y)^3 \, dA$$

where R is the region bounded by the lines $y = 0, x + y = 0$ and $x - y = 2$. Using the substitutions $u = x - y$ and $v = x + y$, evaluate I.

Solution:

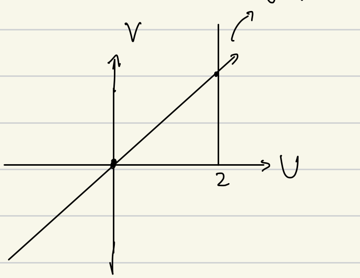
→ Change of variable use jacobian

as $U = x - y$, $V = x + y$

R_1: $y = 0$ $\quad S_1$: $U - V = 0$

R_2: $x + y = 0$ $\quad S_2$: $V = 0$

R_3: $x - y = 2$ $\quad S_3$: $U = 2$



$U = x - y$

$V = x + y$

$U + V = 2x \rightarrow x = \frac{U+V}{2}, \ y = \frac{U-V}{2}$

Jacobian $\rightarrow \begin{vmatrix} \frac{\partial x}{\partial U} & \frac{\partial x}{\partial V} \\ \frac{\partial y}{\partial U} & \frac{\partial y}{\partial V} \end{vmatrix}$

$$= \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} \quad -\frac{1}{4} - \frac{1}{4} = -\frac{2}{4} = -\frac{1}{2} \rightarrow \text{Jacobian} = \left|-\frac{1}{2}\right| = \frac{1}{2}$$

$$= \frac{1}{2} \int_0^2 \int_0^v v r^3 \, dr \, dv \quad x =$$

Jacobian

$$= \int_0^2 v \frac{r^4}{4} \Big|_0^v dv = \frac{1}{4} \int_0^2 v^5 \, dv = \frac{1}{4} \left[\frac{2^6}{6} \right] = \frac{\cancel{64}^{16}}{6} \times \frac{1}{\cancel{4}}$$

$$= \frac{\cancel{16}^{8}}{\cancel{6}_3} = \frac{\cancel{8}^{4}}{3} \times \frac{1}{2} = \frac{4}{3}$$

6A. Use a triple integral in cylindrical coordinates to compute the first moment about the xy-plane (M_{xy}) for the solid in the region bounded above by $z = \sqrt{5 - x^2 - y^2}$ and below by $z = 1$ with the density of the solid given by $\delta(x, y, z) = \sqrt{x^2 + y^2}$.

Moment

Moment = Σ (Mass) (distance)

$= \Sigma \underbrace{(\text{distance})}_{z} \underbrace{(\text{density})}_{r} \underbrace{(\text{Volume})}_{dV}$

Find $M_{xy} = \iiint_R z \delta(x, y, z) \, dA$

$$1^2 = \sqrt{5 - x^2 - y^2}^{\,2}$$

$$1 = 5 - x^2 - y^2$$

$$x^2 + y^2 = 4$$

$$r^2 = 4$$

$$r = 2 \rightarrow 0 \le r \le 2$$

$$= \int_0^{2\pi} \int_0^2 \int_1^{\sqrt{5 - r^2}} z r^2 \, dz \, dr \, d\theta$$

$$= \int_0^{\pi} \int_0^2 \frac{r^2 z^2}{2} \Big|_1^{\sqrt{5 - r^2}} dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 \frac{r^2(5 - r^2)}{2} - \frac{r^2}{2} \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 \frac{5r^2 - r^4 - r^2}{2} \, dr \, d\theta = \int_0^{2\pi} \int_0^2 \frac{4r^2 - r^4}{2} \, dr \, d\theta$$

$$= \pi \left(\frac{4r^3}{3} - \frac{r^5}{5} \Big|_0^2 \right) = \pi \left[\frac{32}{3} - \frac{32}{5} \right] = \pi \left[\frac{160}{15} - \frac{96}{15} \right]$$

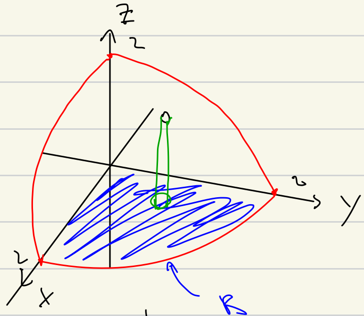
$$= \frac{64\pi}{15}$$

7A. Evaluate the surface integral $\iint_S z\, d(SA)$, where S is the portion of the sphere of radius 2 centered at the origin that lies in the first octant.

find z first and $f(x,y,z)$

$\{x, y, z \geq 0$

Surface Integral (Area) $\Rightarrow \sqrt{(f_x)^2 + (f_y)^2 + 1}$



$$\iint z\, d(SA) =$$

$$= \iint \sqrt{\cancel{4-x^2-y^2}} \cdot \frac{2}{\sqrt{\cancel{4-x^2-y^2}}}\, dA$$

$$= \iint 2\, dA \quad \Big\} \text{ Double Integral = Area of R.}$$

$$= 2\left(\frac{1}{4}\pi(2)^2\right)$$

$$= 2\pi$$

$\left(\frac{1}{4} = \text{first octant}\right)$

$x^2+y^2+z^2=4$ radius $=2$

$z^2 = 4-x^2-y^2$

$z = \sqrt{4-x^2-y^2}$ $\rightarrow f(x,y,z)$

$$f_x = \frac{-x}{\sqrt{4-x^2-y^2}},\quad f_y = \frac{-y}{\sqrt{4-x^2-y^2}}$$

$$(f_x)^2 + (f_y)^2 + 1 = \frac{x^2}{4-x^2-y^2} + \frac{y^2}{4-x^2-y^2} + 1$$

$$\text{Surface Area} = \sqrt{\frac{4}{4-x^2-y^2}}$$

※ Exam 3

1. Find the volume of the solid that lies under the hyperbolic paraboloid $z = 3y^2 - x^2 + 2$ and above the rectangle $R = [-1, 1] \times [1, 2]$ in the xy-plane.

easier than you think

$$\int_{-1}^{1}\int_{1}^{2} 3y^2 - x^2 + 2 \, dy \, dx$$

$$= \int_{-1}^{1} y^3 - x^2 y + 2y \Big|_{1}^{2} \, dx$$

$$= \int_{-1}^{1} (8 - 2x^2 + 4) - (1 - x^2 + 2) \, dx$$

$$= \int_{-1}^{1} 7 - x^2 + 2 \, dx$$

$$= \int_{-1}^{1} 9 - x^2 \, dx = 2\left[9x - \frac{x^3}{3} \Big|_{0}^{1} \right]$$

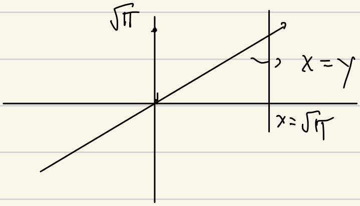
$$= 2\left[9 - \frac{1}{3} \right]$$

$$= 2\left[\frac{26}{3} \right] = \frac{52}{3}$$

//

2. Evaluate the integral by reversing the order of integration.

$$\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \cos(x^2)\,dx\,dy.$$



$$= \int_0^{\sqrt{\pi}} \int_0^{x} \cos(x^2)\,dy\,dx$$

Apply Fubini theorem.

As $f(x,y)$ is continuous on both $0 \le y \le \sqrt{\pi}$ and $y \le x \le \sqrt{x}$

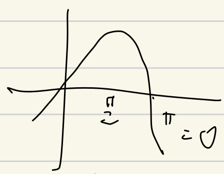
Hence $dy\,dx = dx\,dy$

$$= \int_0^{\sqrt{\pi}} x\cos(x^2)\,dx$$

$u = x^2 \quad du = 2x\,dx$

$dx = \frac{du}{2x}$

Integrate cos.



$$= \frac{1}{2}\left[\sin u \Big|_0^{\pi}\right] = \frac{1}{2}(\sin(\pi)) = \frac{1}{2}(0) = 0 \quad \#$$

3. Let R be the region $R = \{(x, y) | 1 \leq x^2 + y^2 \leq 4, 0 \leq y \leq x\}$. Evaluate the integral by converting to polar coordinates:

$$\iint_R \arctan(y/x)\, dA.$$

Note !!

$\arctan(\tan(\theta))$

$= \theta$ when $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$1 \leq x^2 + y^2 \leq 4$, $0 \leq y \leq x$

$1 \leq r^2 \leq 4$

$1 \leq r \leq 2$

$0 \leq r\sin\theta \leq r\cos\theta$

$r\sin\theta = r\cos\theta$

$\tan\theta = 1$

$\theta = \tan^{-1}(1)$

$\theta = \frac{\pi}{4}$

$0 \leq \theta \leq \frac{\pi}{4}$

$$= \int_0^{\frac{\pi}{4}} \int_1^2 \arctan\left(\frac{\cancel{r}\sin\theta}{\cancel{r}\cos\theta}\right) r\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \int_1^2 \theta\ r\, dr\, d\theta$$

$$= \left[\frac{\theta^2}{2}\Big|_0^{\frac{\pi}{4}}\right]\left[\frac{r^2}{2}\Big|_1^2\right] = \left[\frac{\frac{\pi^2}{16}}{2}\right]\left[2 - \frac{1}{2}\right] = \left[\frac{\pi^2}{32}\right]\left[\frac{3}{2}\right]$$

$$= \frac{3\pi^2}{64}$$

4. Find the volume and centroid of the solid E that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$, using cylindrical or spherical coordinates, whichever seems more appropriate. [Recall that the centroid is the center of mass of the solid assuming constant density.]

Volume : $\underbrace{x^2+y^2+z^2}_{\rho^2} = 1$

$\rho^2 = 1$

$\rho = 1 \to \quad 0 \le \rho \le 1$

find ϕ

$\rho\cos\phi = \sqrt{\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta}$

$\rho\cos\phi = \rho\sin\phi$

$\tan\phi = 1$

$\phi = \frac{\pi}{4} \to \quad 0 \le \phi \le \frac{\pi}{4}$

sphere = $0 \le \theta \le 2\pi$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^1 \rho^2\sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \left[\int_0^{\frac{\pi}{4}} \sin\phi \, d\phi\right]\left[2\pi\right]\left[\frac{1}{3}\right]$$

$$= -\cos\phi \Big|_0^{\frac{\pi}{4}}$$

$$= -\cos\left(\frac{\pi}{4}\right) + \cos(0) = \left(-\frac{1}{\sqrt{2}} + 1\right)(2\pi)\left(\frac{1}{3}\right) =$$

$$= \left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)\frac{2\pi}{3}$$

$$= \frac{2\pi}{3}\left(1 - \frac{1}{\sqrt{2}}\right) \to \frac{2\pi}{3} - \frac{2\pi}{3\sqrt{2}}$$

To calculate Centroid or center of mass of the solid)

$\downarrow$

Since; the region symmetric about the z-axis, therefo $\bar{x} = \bar{y} = 0$
thus we need only to compute the xy-Moment.

$$M_{xy} = \iiint_E \underbrace{z \, dV}_{(\text{distance})(\text{density})} = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^1 \rho\cos\phi \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 2\pi \left[\int_0^{\frac{\pi}{4}} \cos\phi \sin\phi \, d\phi \right] \left[\frac{1}{\cancel{4}\,2} \right]$$

$$= \frac{\cancel{2}\pi}{2} \left[\frac{u^2}{2} \Big|_0^{\frac{1}{\sqrt{2}}} \right] = \frac{\pi}{2} \left[\frac{\frac{1}{2}}{2} \right]$$

$$\bar{z} = \frac{\pi}{8}$$

So centroid = $\left(0, 0, \dfrac{\frac{\pi}{8}}{\frac{2\pi}{3}\left(1 - \frac{1}{\sqrt{2}}\right)} \right)$

5. Let R be the parallelogram with vertices $(-1,3), (1,-3), (3,-1)$, and $(1,5)$. Use the transformation $x = \frac{1}{4}(u+v), y = \frac{1}{4}(v-3u)$ to evaluate the integral

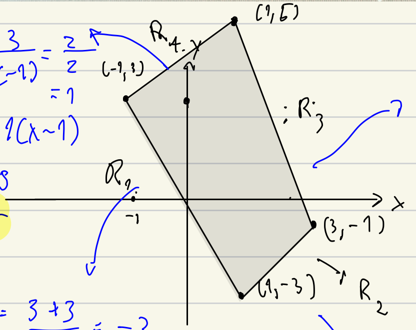
$$\iint_R (4x + 8y)\, dA.$$

$\frac{5-3}{1-(-1)} = \frac{2}{2} = 1$

$y - 5 = 1(x-1)$

$y = x - 1 + 5$

$y = x + 4$



$\frac{5-(-1)}{1-3} = \frac{6}{-2} = -3$

$y - 5 = -3(x-1)$

$y - 5 = -3x + 3$

$y = -3x + 8$

$R_1 = \frac{3+3}{-1-1} = -3$

$y = -3x$

$R_2 : \frac{-1+3}{3-1} = \frac{2}{2} = 1$

$y - (-1) = 1(x-3)$

$y + 1 = x - 3$

$y = x - 4$

$R_1 : \quad y + 3x = 0$

$R_2 : \quad x - y = 4$

$R_3 : \quad y + 3x = 8$

$R_4 : \quad x - y = -4$

as $x = \frac{1}{4}(u+v)$

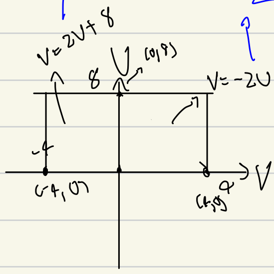
and $y = \frac{1}{4}(v - 3u)$

$u = \frac{v-8}{2}$

$v = 2u + 8$

$u = \frac{8-v}{2}$

$v = -2u + 8$



1st case

$4x = u + v$

$-(4y = -3u + v)$

$4x - 4y = 4u$

$x - y = u$

$S_1 : \quad v = 0$

$S_2 : \quad u = 4$

$S_3 : \quad v = 8$

$S_4 : \quad u = -4$

2nd Case.

$3(4x = u + v)$

$4y = -3u + v$

$12x = 3u + 3v$

$4y = -3u + v$

$12x + 4y = 4v$

$3x + y = v$

As it is one-to-one and Dg is continuous and not vanish; we can apply Jacobian transformation.

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{4} & \frac{1}{4} \end{vmatrix} = \frac{1}{16} - \left(-\frac{3}{16}\right) = \frac{4}{16} = \frac{1}{4}$$

$$= \int_0^8 \int_{-4}^{4} u + v + 2v - 6u \; du\, dv \quad \Big\}$$ Fubini apply as it continuous over the region for $f(x,y)$

Jacobian

$$= \int_0^8 \int_{-4}^{4} 3v - 5u \left|\frac{1}{4}\right| du\, dv$$

Not even function, cannot change to $2\int_0^4$

$$= \frac{1}{4}\int_0^8 3vu - \frac{5u^2}{2} \Big|_{-4}^{4}$$

$$= \frac{1}{4}\int_0^8 \left[(12v - 40) - (-12v - 40)\right) dv$$

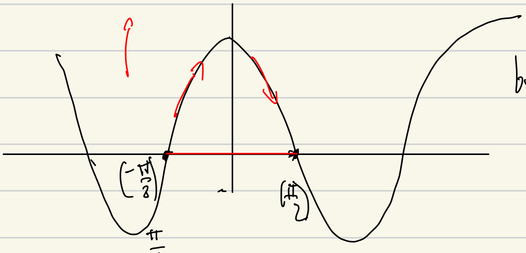
$$= \frac{1}{4}\int_0^8 24v \; dv$$

$$= \frac{1}{4}\left[12v^2 \Big|_0^8 \right] = \frac{1}{4}\left[12(8)^2 \right] = 192 \;\#$$

6. Use Green's Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle e^{-x} + y^2, e^{-y} + x^2 \rangle$, and where C consists of the arc of the curve $y = \cos x$ from $(-\pi/2, 0)$ to $(\pi/2, 0)$ and the line segment from $(\pi/2, 0)$ to $(-\pi/2, 0)$.

P Q

clockwise = negative



simple close curve, piecewise smooth, but negative oriented (clockwise) hence we can apply green theorem

$$= -\int \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA$$

$$-\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\cos x} 2x - 2y \, dy \, dx$$

$$= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2xy - y^2 \Big|_0^{\cos x} dx$$

$$= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2x\cos x - \cos^2 x \, dx$$

$$= -2\left[\int_0^{\frac{\pi}{2}} 2x\cos x \, dx - \int_0^{\frac{\pi}{2}} \cos^2 x \, dx\right]$$

= 0 (odd function)

$$-2\left[-\left(\frac{1}{2}x + \frac{1}{4}\sin(2x)\right)\Big|_0^{\frac{\pi}{2}}\right]$$

$$-2\left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$ //

7. Find the curl and divergence of the vector field $\mathbf{F}$. If it is conservative, find a function f such that $\mathbf{F} = \nabla f$.

$$\mathbf{F}(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}\langle x, y, z\rangle.$$

Curl = $\nabla \times F$, Divergence = $\nabla \cdot F$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{z}{\sqrt{x^2+y^2+z^2}} \end{vmatrix}$$

The vector of form $F = \frac{r}{\rho}$ is conservative

and if conservative: Curl F need to be $= 0$.

$= 0 - 0 - 0$ $\quad \nabla \times F = \langle 0, 0, 0 \rangle$

Hence it is conservative.

$$\boxed{\text{Div } F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}}$$

Def: The tendency of the fluid to leave the point.

$$= \frac{2x^2}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \frac{2y^2}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \frac{2z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$= \frac{2}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{2}{\rho}$$

f function $= \sqrt{x^2+y^2+z^2} + C$

8. Find the surface area of the surface defined parametrically by the vector equation $\mathbf{r}(u,v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + v\mathbf{k}, 0 \le u \le 1, 0 \le v \le u$.

6.4 Surface Integrals

Theorem 6.11 (**Surface Area**). *Let a surface S is given by an equation $\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v))$ for $(u,v) \in D$, then the surface area of S is*

$$A(S) = \iint_D \|\mathbf{r}_u \times \mathbf{r}_v\|\, dA.$$

Definition 6.12 (**Surface Integrals of Scalar Functions**). The surface integral of $f : \mathbb{R}^3 \to \mathbb{R}$ on the surface S with equation $\mathbf{r}(u,v)$ for $(u,v) \in D$ with continuous component functions and non-zero $\mathbf{r}_u, \mathbf{r}_v$ is defined as

$$\iint_S f(x,y,z)\, dS = \iint_D f(\mathbf{r}(u,v))\|\mathbf{r}_u \times \mathbf{r}_v\|\, dA.$$

$$\iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA.$$

Surface integral "Vector Field"

$$r_u = \langle \cos(v), \sin(v), 0 \rangle$$

$$r_v = \langle -u\sin(v), u\cos(v), 1 \rangle$$

$$r_u \times r_v$$

$$= \begin{vmatrix} i & j & k \\ \cos(v) & \sin(v) & 0 \\ -u\sin(v) & u\cos(v) & 1 \end{vmatrix}$$

$$= i(\sin(v) - 0) + j(0 - \cos(v)) + k(u\cos^2(v) + u\sin^2(v))$$

$$= \langle \sin(v), -\cos(v), u \rangle$$

$$\|r_u \times r_v\| = \sqrt{\sin^2(v) + \cos^2(v) + u^2} = \sqrt{1+u^2}$$

$$\text{Area} = \int_0^1 \int_0^u \sqrt{1+u^2}\, dv\, du$$

$$= \int_0^1 u\sqrt{1+u^2}\, du$$

$$U = 1+u^2 \quad dU = 2u\, du$$

$$= \frac{1}{2}\int_0^1 \sqrt{U}\, du \rightarrow \frac{1}{2}\left[\frac{2U^{\frac{3}{2}}}{3}\Big|_1^2\right]$$

$$= \frac{1}{2}\left[\frac{2(2)^{\frac{3}{2}}}{3} - \frac{2}{3}\right]$$

$$= \frac{1}{2}\left[\frac{2(2)^{\frac{3}{2}} - 2}{3}\right] \quad \#$$

Find the work done by the force field to move an object from P to Q

$F(x,y) = 2y^{\frac{3}{2}}\,i + 3x\sqrt{y}\,j$, $P(1,1)$ to $Q(2,4)$.

$W = \int F \cdot dr$ let $F(P,Q) = \langle 2y^{\frac{3}{2}}, 3x\sqrt{y} \rangle$

Check if F is conservative such that $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$

$= 3\sqrt{y} \neq \frac{4y^{\frac{5}{2}}}{5}$

$F = \nabla f$ not conservative, $!= 0$.

$\int_x f_x = 2xy^{\frac{3}{2}} + g(y)$, $\int_y f_y = 2xy^{\frac{3}{2}} + g_x(y)$

$f(x,y) = 2xy^{\frac{3}{2}} + K$

From fundamental theorem of line integral, as $f(x,y)$ is continuous over P to Q then $\int F \cdot dr = f(r(b)) - f(r(a))$

$$= 2xy^{\frac{3}{2}} + K \Big|_{(1,1)}^{(2,4)}$$

$$= 4(4^{\frac{3}{2}}) + K - (2(1^{\frac{3}{2}}) + K)$$

factor out exponential ให้เหลือแค่ root.

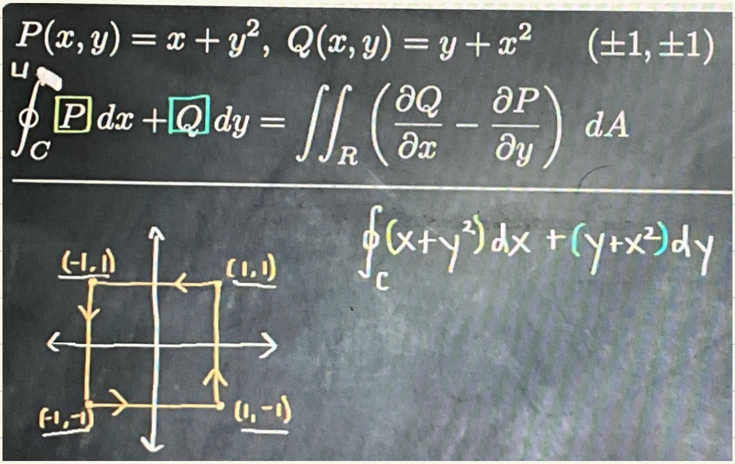
$$4(4^{\frac{1}{2}})^3 + K - (2(1^{\frac{1}{2}})^3 + K$$

$$32 + K - 2 + K$$

$$= 30 \ \#$$

Note: To use FTLI need to have potential function

Green Theorem.



Simple close curve, piecewise smooth and also positive orientation (counter clockwise), Hence we can apply green theorem.

$$\iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$\int_{-1}^{1}\int_{-1}^{1} 2x - 2y \; dy\,dx$$

$$= \int_{-1}^{1} 2xy - y^2 \Big|_{-1}^{1} = \int_{-1}^{1} (2x-1) - (-2x-1) = \int_{-1}^{1} 4x \, dx$$

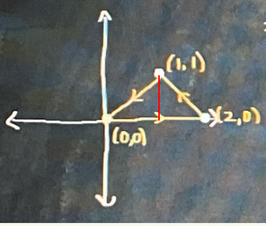
$$= 2x^2 \Big|_{-1}^{1} = 2 - (2) = 0$$

$\boxed{P(x,y)} = y + e^x$, $\boxed{Q(x,y)} = 2x^2 + \cos y$ $\quad (0,0),\ (1,1),\ (2,0)$

$$\left[\oint_C \boxed{P}\,dx + \boxed{Q}\,dy\right] = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Simple close curve, positive oriented and piecewise smooth such that we can apply green theorem.

$$\Rightarrow \oint_C \underbrace{\boxed{y+e^x}}_{P}\,dx + \underbrace{\boxed{2x^2+\cos y}}_{Q}\,dy$$



$\rightarrow \dfrac{1-0}{1-2} = \dfrac{1}{-1} = -1,$

$y-1 = -1(x-1) \quad y-1 = -x+1$

$y = -x+2$

$$\frac{\partial Q}{\partial x} = 4x, \quad \frac{\partial P}{\partial y} = 1$$

$$= \iint 4x-1 \; dA$$

$$\int_0^1\int_0^x 4x-1\,dy\,dx + \int_1^2\int_0^{-x+2} 4x-1\,dy\,dx$$

$$= \int_0^1 (4x-1)x\,dx \qquad \int_1^2 (4x-1)(-x+2)\,dx$$

$$= \int_0^1 4x^2 - x\,dx \qquad = \int_1^2 -4x^2+8x+x-2\,dx$$

$$= \frac{4x^3}{3} - \frac{x^2}{2}\Big|_0^1 = \frac{4}{3} - \frac{1}{2} \qquad = \int_1^2 -4x^2+9x-2\,dx$$

$$= \frac{5}{6} + \qquad = -\frac{4x^3}{3} + \frac{9x^2}{2} - 2x\Big|_1^2$$

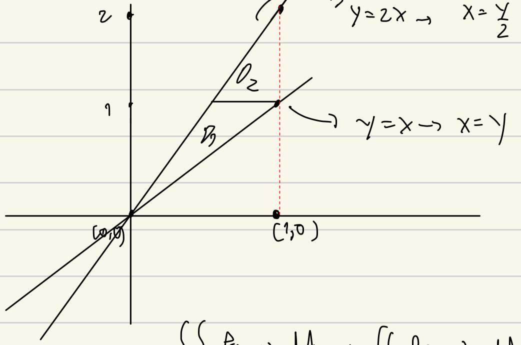
$$= -\frac{32}{3} + \frac{36}{2} - 4 - \left(-\frac{4}{3} + \frac{9}{2} - 2\right)$$

$= \dfrac{5}{6} +$ (DIY) $\rightarrow$ Ans $= \dfrac{28}{6} \longrightarrow 3$

Exam 3 MIT

Problem 1. a) Draw a picture of the region of integration of $\int_0^1 \int_x^{2x} dy dx$.

b) Exchange the order of integration to express the integral in part (a) in terms of integration in the order $dx dy$. Warning: your answer will have two pieces.

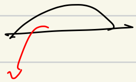


$$\iint_{D_1} f(x,y)\, dA + \iint_{D_2} f(x,y)\, dA$$

$$= \int_0^1 \int_{\frac{y}{2}}^{y} f(x,y)\, dx dy + \int_1^2 \int_{\frac{y}{2}}^{1} f(x,y)\, dx dy.$$

Problem 2. a) Find the mass M of the upper half of the annulus $1 < x^2 + y^2 < 9$ ($y \geq 0$) with density $\delta = \frac{y}{x^2+y^2}$.

b) Express the x-coordinate of the center of mass, $\bar{x}$, as an iterated integral. (Write explicitly the integrand and limits of integration.) Without evaluating the integral, explain why $\bar{x} = 0$.



a.) $M = \iint_R \delta(x,y)\, dA$ as $1 < x^2+y^2 < 9$

We can use polar coordinate.

$1 < r^2 < 9$

$1 < r < 3$

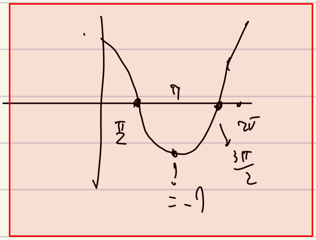
as $y \geq 0$ $\quad 0 \leq \theta \leq \pi$

$$\int_0^{\pi} \int_1^3 \frac{\cancel{r}\sin\theta}{\cancel{r^2}} \cancel{r} dr\, d\theta$$

$$= \int_0^{\pi} \int_1^3 \sin\theta\, dr\, d\theta$$

Both limit is constant and $f(x,y)$ is continuous over the region hence we can apply Fubini theorem.

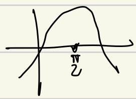
$$= \left[\int_0^{\pi} \sin\theta \, d\theta\right]\left[\int_1^3 1 \, dr\right]$$



$$= \left[-\cos(\pi) + 1\right]\left[\, r \Big|_1^3 \,\right]$$

$$\left[-(-1)+1\right]\left[\, 2 \,\right]$$

$$= (2)(2) = 4$$



6.) $\tilde{x} = \dfrac{M_x}{M} \to M_x = \iint_R x \, \rho(x,y) \, dA$

$$\int_0^{\pi}\int_1^3 r\sin\theta \, r\cos\theta$$

equal to 0 as it is odd function and $\sin(\pi) = 0$ as well as $\sin(0)$

Hence $\tilde{x} = \dfrac{0}{M} = 0$ ∎ Proven.

$P \quad Q$

Problem 3. a) Show that $\mathbf{F} = (3x^2 - 6y^2)\hat{\imath} + (-12xy + 4y)\hat{\jmath}$ is conservative.

b) Find a potential function for $\mathbf{F}$.

c) Let C be the curve $x = 1 + y^3(1-y)^3$, $0 \le y \le 1$. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

ได้ สม. และ p.n.t มา ใช้ sub p.n.t y or x เพื่อหา p.n.t รวม

a.) Conservative if $F = \nabla f$ such that

$$\left.\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}\right\} = -12y = -12y$$

equal hence conservative

b.) $F = \nabla f$ let $F(P, Q) = \langle 3x^2 - 6y^2, -12xy + 4y \rangle$

$$\int_x f_x = x^3 - 6xy^2 + g(y) \qquad \int_y f_y = -6xy^2 + 2y^2 + g_x(y)$$

$$f(x,y) = x^3 + 2y^2 - 6xy^2 + K$$

c.) According to Fundamental theorem of calculus as $f(x,y)$ is continuos over $[0, \le y \le 1]$

Hence we need to find value when it start and stop -

eq: $x = 1 + y^3(1-y)^3 \Rightarrow$ when $y = 0 \Rightarrow x = 1$

$y = 1 \rightarrow x = 1$

Hence point from $(1, 0)$ to $(1, 1)$

$$= x^3 + 2y^2 - 6xy^2 \Big|_{(1,0)}^{(1,1)} = (3-6) - (1)$$

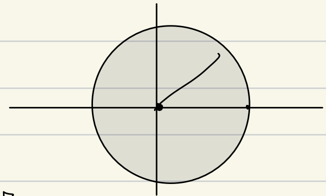
$$= -4$$

Check later

Problem 4. a) Express the work done by the force field $\mathbf{F} = (5x+3y)\hat{\imath} + (1+\cos y)\hat{\jmath}$ on a particle moving counterclockwise once around the unit circle centered at the origin in the form $\int_a^b f(t)dt$. (Do not evaluate the integral; don't even simplify $f(t)$.)

b) Evaluate the line integral using Green's theorem.

$W = \int F \cdot dr$



a.) $F = \nabla f$

let $F(P,Q) = \langle 5x+3y,\ 1+\cos y \rangle$ $\qquad \int_a^b f(t)\,dt$

$\int F \cdot dr = P\,dx + Q\,dy$

let $r(t) =$ $x = \overset{r=1}{\cos t}$ $\quad y = \overset{r=1}{\sin t}$ $\quad \theta \Rightarrow 0 \le \theta \le 2\pi$

$dx = -\sin t\,dt \quad dy = \cos t\,dt$

$$W = \int_0^{2\pi} (5(\cos t) + 3\sin t)(-\sin t)\,dt + (1 + \cos(\sin(t)))\cos(t)\,dt$$

b.) Simple closed curve; Piecewise smooth and positive oriented (counter clockwise)

Hence we can apply green theorem.

$$\iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

$$\int_0^{2\pi}\int_0^1 (0 - 3)\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 -3r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[-\frac{3r^2}{2} \Big|_0^1 \right] d\theta$$

$$= \int_0^{2\pi} -\frac{3}{2} \, d\theta$$

$$= -\frac{3}{2} \left[\theta \Big|_0^{2\pi} \right]$$

$$= -\frac{3}{\cancel{2}} \times \cancel{2}\pi = -3\pi$$ #

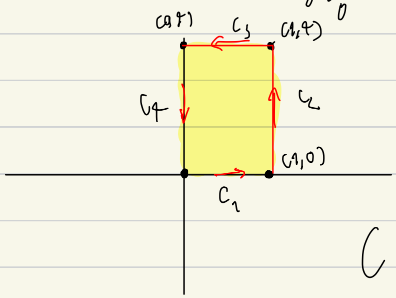
Problem 5. Consider the rectangle R with vertices $(0,0)$, $(1,0)$, $(1,4)$ and $(0,4)$. The boundary of R is the curve C, consisting of C_1, the segment from $(0,0)$ to $(1,0)$, C_2, the segment from $(1,0)$ to $(1,4)$, C_3 the segment from $(1,4)$ to $(0,4)$ and C_4 the segment from $(0,4)$ to $(0,0)$. Consider the vector field

$$\mathbf{F} = (xy + \sin x \cos y)\hat{\mathbf{i}} - (\cos x \sin y)\hat{\mathbf{j}}$$

a) Find the flux of $\mathbf{F}$ out of R through C. Show your reasoning.

b) Is the total flux out of R through C_1, C_2 and C_3, more than, less than or equal to the flux out of R through C? Show your reasoning.

a.) Surface Integral $= \iint_D F \cdot (r_u \times r_v)\, dA$.



Cheat sheet (Flux).

$$\boxed{\oint_C F \cdot \hat{n}\, ds = \iint_R \text{div}\, F \, dx\, dy}$$

$$\left(\frac{d}{dx}(xy + \sin x \cos y) - \frac{d}{dy}(\cos x \sin y) \right)$$

$$\iint_R (y + \cos x \cos y - \cos x \cos y)\, dx\, dy = \iint_R y \, dx\, dy$$

$$= \int_0^4 \int_0^1 y \, dx \, dy = \frac{y^2}{2}\Big|_0^4 = 8 \quad \#$$

6. On C_4, $x=0$, so $F = -\sin y \, \hat{j}$, where $\hat{n} = -\hat{i}$, Hence $F \cdot \hat{n} = 0$

Therefore: the Flux of F through C_4 equals 0.

Thus
$$\int_{C_1+C_2+C_3} F \cdot \hat{n} \, ds = \oint_C F \cdot \hat{n} \, ds - \underbrace{\int_{C_4} F \cdot \hat{n} \, ds}_{=0} = \oint_C F \cdot \hat{n} \, ds;$$

Hence:
$$\int_{C_1+C_2+C_3} F = \oint_C F \cdot \hat{n} \, ds$$

Problem 6. Find the volume of the region enclosed by the plane $z = 4$ and the surface
$$z = (2x - y)^2 + (x + y - 1)^2.$$
(Suggestion: change of variables.)

let $U = 2x - y$, $V = x + y - 1$

1st case

$$U = 2x - y$$
$$V = x + y - 1$$
$$U + V = 3x - 1$$
$$U + V + 1 = 3x \rightarrow x = \frac{U+V+1}{3}$$

$$U = 2x - y$$
$$-2\,(V = x + y - 1)$$
$$U = 2x - y$$
$$-2V = -2x - 2y + 2$$
$$U - 2V = -3y + 2$$
$$U - 2V - 2 = -3y$$
$$y = \frac{2 + 2V - U}{3}$$

Hence as $z = U^2 + V^2$

$$z = r^2$$
$$4 = r^2$$
$$r = 2 \rightarrow 0 \le r \le 2, \quad \theta \rightarrow 0 \le \theta \le 2\pi$$

One to one and continuous, hence can apply Jacobian

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{vmatrix} = \frac{2}{9} - \left(-\frac{1}{9}\right) = \frac{3}{9} = \frac{1}{3}$$

As $z = u^2 + v^2$

$4 - u^2 - v^2 = 0$

$\downarrow$

volume $= 4 - r^2$

$$\int_0^{2\pi}\int_0^2 (4-r^2)\,\frac{1}{3}r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{2}{3}r^2 - \frac{1}{12}r^4\right]_0^2 d\theta$$

Ans $= \boxed{\frac{8\pi}{3}}$

In double integral; volume need function inside not just 1

Exam 4 # MIT

Problem 1. (10 points)

Let C be the portion of the cylinder $x^2+y^2 \leq 1$ lying in the first octant ($x \geq 0$, $y \geq 0$, $z \geq 0$) and below the plane $z = 1$. Set up a triple integral in *cylindrical coordinates* which gives the moment of inertial of C about the z-axis; assue the density to be $\delta = 1$.
(Give integrand and limits of integration, but *do not evaluate*.)

$$\int_0^{\frac{\pi}{2}}\int_0^1\int_0^1 \underbrace{(x^2+y^2)}_{r^2}\,1\,r\,dz\,dr\,d\theta$$

#

Problem 2. (20 points: 5, 15)

a) A solid sphere S of radius a is placed above the xy-plane so it is tangent at the origin and its diameter lies along the z-axis. Give its equation in *spherical coordinates.*

b) Give the equation of the horizontal plane $z = a$ in spherical coordinates.

c) Set up a triple integral in spherical coordinates which gives the volume of the portion of the sphere S lying *above* the plane $z = a$. (Give integrand and limits of integration, but *do not evaluate.*)

plane use $\sec\phi$

a). tangent at the origin: Hence Coordinate $= (0,0,a)$

$$= \boxed{x^2+y^2+(z-a)^2 = a^2}$$

$$\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta + (\rho\cos\phi - a)^2 = a^2$$

$$\rho^2\sin^2\phi + (\rho\cos\phi - a)^2 = a^2$$

$$\rho^2\sin^2\phi + \rho^2\cos^2\phi - 2a\rho\cos\phi + \cancel{a^2} = \cancel{a^2}$$

$$\rho^2 - 2a\rho\cos\phi = 0$$

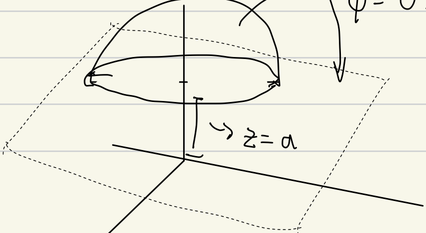
$$\rho(\rho - 2a\cos\phi) = 0$$

$$\rho = 0,\ 2a\cos\phi$$

b.) For plane use as $z=0 \rightarrow \rho\cos\phi = a \rightarrow \rho = \dfrac{a}{\cos\phi} \rightarrow \rho = a\sec\phi$

c.) =

$\phi = 0 \leq \phi \leq \frac{\pi}{4}$!!!



Triple integral $$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_{a\sec}^{2a\cos\phi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

Problem 3. (20 points: 5, 15)

Let $\vec{F} = (2xy + z^3)\,\hat{\imath} + (x^2 + 2yz)\,\hat{\jmath} + (y^2 + 3xz^2 - 1)\,\hat{\mathbf{k}}$.

a) Show $\vec{F}$ is conservative.

b) Using a systematic method, find a potential function $f(x, y, z)$ such that $\vec{F} = \vec{\nabla} f$. Show your work even if you can do it mentally.

a) Conservative if $\nabla \times F = 0$ (Curl Definition)

$$\begin{vmatrix} j & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy+z^3 & x^2+2yz & y^2+3xz^2-1 \end{vmatrix}$$

$$= \langle \underbrace{(2y-2y)}_{0}\hat{i},\ \underbrace{(3z^2-3z^2)}_{0}\hat{j},\ \underbrace{(2x-2x)}_{0}\hat{k} \rangle$$

Hence it is Conservative . .

b.) $\vec{F} = \nabla f \rightarrow F(P,Q,Z) = \langle (2xy+z^3)\hat{i}, (x^2+2yz)\hat{j}, (y^2+3xz^2-1)\hat{k} \rangle$

$$\int_x f_x = \cancel{x^2 y} + \cancel{z^3 x}, \quad \int_y f_y = \cancel{x^2 y} + \cancel{y^2 z}, \quad \int_z f_z = \cancel{y^2 z} + \cancel{x z^3} - z$$

$$f(x,y,z) = x^2 y + y^2 z + x z^3 - z + K \ \#$$

Problem 4.(25 points: 15, 10)

Let S be the surface formed by the part of the paraboloid $z = 1 - x^2 - y^2$ lying above the xy-plane, and let $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath} + 2(1-z)\,\hat{\mathbf{k}}$.

Calculate the flux of $\vec{F}$ across S, taking the upward direction as the one for which the flux is positive. Do this in two ways:

a) by direct calculation of $\int\int_S \vec{F}\cdot\hat{\mathbf{n}}\,dS$;

b) by computing the flux across a simpler surface and using the divergence theorem.

a) $\iint_S \vec{F}\cdot\hat{n}\,ds$, $z = 1 - x^2 - y^2$

$\hat{n}\,dS = \langle -f_x, -f_y, 1\rangle\,dA$ } Only direction.

$\hat{n} = \langle 2x, 2y, 1\rangle$

$= \iint \langle x, y, 2(1-z)\rangle \cdot \langle 2x, 2y, 1\rangle\,dA$

$= \iint_S 2x^2 + 2y^2 + 2(1-z)\,dA$

$z = 1 - x^2 - y^2$

$= \iint_S 2x^2 + 2y^2 + 2(\cancel{1} - \cancel{1} + x^2 + y^2)\,dA$

$= \iint_S 4x^2 + 4y^2\,dA$

$= \int_0^{2\pi}\int_0^1 4r^2 r\,dr\,d\theta$ ※ [DIY] $= 2\pi$

※ Final Exam cumulative, HKU

8. Find the local max, min, and saddle points for $f(x,y) = \frac{1}{3}x^3 + \frac{1}{3}y^3 - xy + 4$ (if any exist).

Polynomial, Hence continuous and we can find Partial derivatives

$Df(x,y) = \quad f_x = x^2 - y = 0 \qquad f_y = y^2 - x = 0$

$x^2 = y \qquad y^2 = x$

$(x^2)^2 = x$

$x^4 = x \rightarrow \quad x^4 - x = 0$

$x(x^3 - 1) = 0$

$x = 0, \quad x^3 = 1$

$x = 1$

when $x = 1 \rightarrow y = 1$, when $x = 0$, $y = 0$

Hence crit p.n.t: $(1,1)$, $(0,0)$

Second Derivative test.

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2x & -1 \\ -1 & 2y \end{vmatrix} = 4xy - 1$$

$f(x,y)$

Check ที่ eq ของ Second derivative test ก่อน

at p.n.t $(1,1)$ $\quad f_{xy}(x,y) \Rightarrow (1,1) = 4 - 1 = 3 > 0$

ถ้ามากกว่า 0 $\rightarrow$ check f_{xx}

<0 saddle p.n.t

$f_{xx} = (1,1) = 2 > 0$

Hence $(1,1)$ is local Min

at p.n.t $(0,0)$ $\quad f(0,0) = -1 < 0$

Hence it is saddle p.n.t

9. Find the local max, min, and saddle points for $f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2$.

Polynomial hence continuous

$f_x = 6x^2 + y^2 + 10x \rightsquigarrow f_x = 0$

$6x^2 + y^2 + 10x = 0$

$f_y = 2xy + 2y = 0$

$2y(x+1) = 0$

$2y = 0 \quad , x = -1$

$\boxed{\text{Point (1)} = (-1,0),\ (-1,2),\ (-1,-2),\ (0,0),\ (-\tfrac{5}{3},0)}$

when $x = -1 \rightarrow \quad 6 + y^2 - 10 = 0$

$y^2 = 10 - 6$

$y^2 = 4$

$y = \pm 2$

when $y = 0 \quad 6x^2 + 10x = 0$

$2x(3x+5) = 0$

$x = 0, \quad x = -\frac{5}{3}$

Second Derivative Test, $\quad f_{xx} = 12x + 10,\ f_{yy} = 2x + 2,\ f_{xy} = 2y,\ f_{yx}\ 2y$

$$\begin{vmatrix} 12x+10 & 2y \\ 2y & 2x+2 \end{vmatrix} = (12x+10)(2x+2) - (2y)(2y)$$

$= 24x^2 + 24x + 20x + 20 - 4y^2$

$= 24x^2 + 44x + 20 - 4y^2$

P.n.t $(-1,0) = 24-44+20 = 0$ → Inconclusive

not use.

$f_{xy}(-1,2) = 24-44+20-16 = -16 < 0$

↓

$(-1,2)$ is Saddle p.n.t.

$f(-1,-2) = -16$ also saddle p.n.t

$f(0,0) = 20 > 0$ $\quad f_{xx}(0,0) = 10 > 0$

↓

Hence its local Min

At $(-\frac{5}{3},0) = \frac{100}{3} > 0$ → $f_{xx}(-\frac{5}{3},0) = -10 < 0$

↓

Hence ⇒ local Max

10. Use Lagrange Multipliers to find the maximum and minimum of $f(x,y) = x^2y$ subject to the constraint $x^2 + y^2 = 1$

optimize $= x^2y$, Constraint $x^2+y^2=1$

As $f(x,y)$ is polynomial hence it is continuous and we can find the partial derivative.

Check if ∇g vanish over f or not

$\nabla g = \langle 2x, 2y \rangle \neq 0$

↓

Hence we can apply lagrange multiplier

$= \nabla f(x,y,z) = \lambda \nabla g(x,y,z)$

$\langle 2xy, x^2 \rangle = \lambda \langle 2x, 2y \rangle$

↓

$\cancel{2x}\lambda = \cancel{2x}y \longrightarrow \lambda = y$ Better way

$2x\lambda - 2xy = 0$

$2x(x-y) =$

$\boxed{x=0,\ \lambda=y}$

$2y\lambda = x^2 \longrightarrow 2y^2 = x^2$

$x^2+y^2=1 \rightarrow (3)$

$y^2 = \frac{x^2}{2} \rightarrow$ plug to (3)

$x^2 + \frac{x^2}{2} = 1$

$\frac{3}{2}x^2 = 1$

$x^2 = \frac{2}{3}$

$x = \pm\sqrt{\frac{2}{3}}$

case $x=0$

2 c.p. $(0,1)$ and $(0,-1)$

Case 1 when $x = \sqrt{\frac{2}{3}} \Rightarrow \frac{2}{3} + y^2 = 1$

$y^2 = 1 - \frac{2}{3}$

$y^2 = \frac{1}{3}$

$y = \pm\frac{1}{\sqrt{3}}$

Hence p.n.t $= \left(\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}\right), \left(\sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{3}}\right)$

Case 2 when $x = -\sqrt{\frac{2}{3}}$ = same

$f\left(\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}\right) = x^2y = \left(\sqrt{\frac{2}{3}}\right)^2 \frac{1}{\sqrt{3}} = \frac{2}{3}\times\frac{1}{\sqrt{3}} = \frac{2}{3\sqrt{3}}$ $\rightarrow$ local Max

$f\left(\sqrt{\frac{2}{3}}, \frac{-1}{\sqrt{3}}\right) = \left(\sqrt{\frac{2}{3}}\right)^2 \frac{-1}{\sqrt{3}} = \frac{2}{3}\times\frac{-1}{\sqrt{3}} = \frac{-2}{3\sqrt{3}}$ $\rightarrow$ local Min

12. Setup the triple integral in the order of $dz\,dx\,dy$ and again as $dz\,dy\,dx$ to find the volume of the solid tetrahedron which is bounded by $3x+y+z=1$ and the coordinate planes (i.e., the first octant).

$3x+y+z=1$

when $z=0$

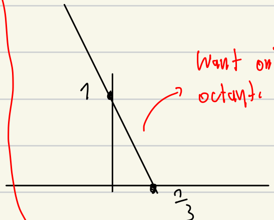
$3x+y=1$

$y=1-3x$

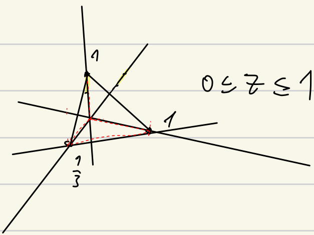
when $y=0$ $\quad 0=1-3x$

$3x=1$

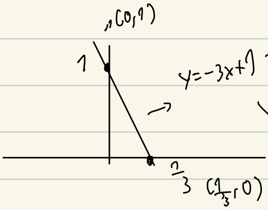
$x=\frac{1}{3}$



want only first octant.



$0 \le z \le 1-3x-y$



$y=-3x+1$ $\quad \frac{1-0}{0-\frac{1}{3}} = \frac{1}{-\frac{1}{3}} = -3$

$y=-3x+1$

$3x=1-y$

$x=\frac{1-y}{3}$

Type 1: $dy\,dx$

$$\#1 = \int_0^{\frac{1}{3}} \int_0^{-3x+1} \int_0^{1-3x-y} f(x,y,z)\, dz\, dy\, dx \quad \#$$

$$\#2 = \int_0^{1} \int_0^{\frac{1-y}{3}} \int_0^{1-3x-y} f(x,y,z)\, dz\, dx\, dy$$

Integral อันนอก (ใส่ใน)

\# ต้องมี variable x หรือ y

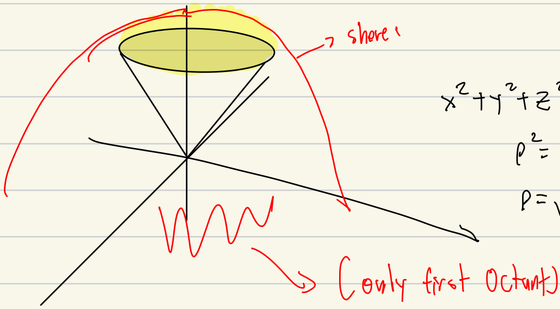
หรือทั้งสองอย่าง

ห้ามเป็น constant.

13. Set up, do not evaluate the triple integral in rectangular, cylindrical, and spherical coordinates to find the volume of the solid in the first octant bounded above by $x^2 + y^2 + z^2$ $= 12$ and bounded below by $z = \sqrt{x^2 + y^2}$.

Note: $z = \sqrt{x^2+y^2}$ is cone at $\phi = \frac{\pi}{4}$

Note: Rectangular interm of x-y.



$x^2+y^2+z^2 = 12$

$\rho^2 = 12$

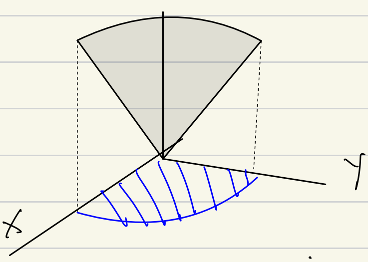
$\rho = \sqrt{12}$

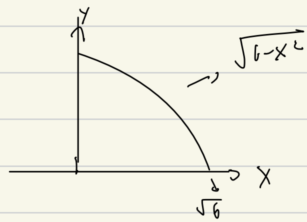
For Rectangular, Make that all function are interm of x and y.

Top functio : $z = \sqrt{12 - x^2 - y^2}$

Bottom Function : $z = \sqrt{x^2+y^2}$

Next Find Projection onto x-y plane to Find D.





set $z = \sqrt{12 - x^2 - y^2} = z = \sqrt{x^2+y^2}$

$12 - x^2 - y^2 = x^2 + y^2$

$2x^2 + 2y^2 = 12$

$x^2 + y^2 = 6 \longrightarrow y^2 = 6 - x^2$

$y = \sqrt{6 - x^2}$

Rectangle

$$= \int_0^{\sqrt{6}} \int_0^{\sqrt{6-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{12-x^2-y^2}} f(x,y,z)\, dz\, dy\, dx$$

$x^2+y^2=6 \quad r^2=6 \quad r=\sqrt{6}$

Cylindricle $\Big\} = \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{6}} \int_r^{\sqrt{12-r^2}} r \, dz \, dr \, d\theta$

Problem stated only first octant

Sphericle $= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{12}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

14. Find the Jacobian for the transformation $x = u^2v + v^2$ and $y = uv^2 - u^2$.

Jacobian = as it is one-to-one, polynomial ∴ Hence it is continuous and we can find it first partial derivative

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial y} \end{vmatrix} = \begin{vmatrix} 2uv & u^2+2v \\ v^2-2u & 2vu \end{vmatrix} = 4u^2v^2 - \left[(u^2+2v)(v^2-2u)\right]$$

$$= 4u^2v^2 - \left[u^2v^2 - 2u^3 + 2v^3 - 4vu\right]$$

$$= 3u^2v^2 - 2u^3 + 2v^3 - 4vu$$

//

15. Evaluate $\iint_D (3x-y)^{3/2}(x+y)^5\, dA$ where D is the region bounded by $y=-x$, $y=-x+1$, $y=3x$, and $y=3x-1$. Use the change of variables $u=3x-y$ and $v=x+y$.

$y=3x$

as $U = 3x - y,\ V = x + y$

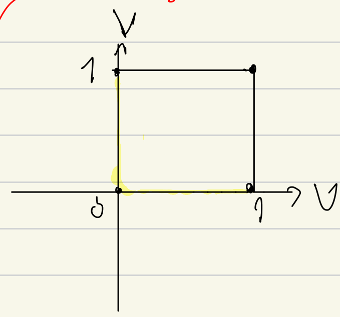
$R_1:\ x+y=0$ $\quad S_1:\ V=0$

$R_2:\ x+y=1$ $\quad S_2:\ V=1$

$R_3:\ 3x-y=0$ $\quad S_3:\ U=0$

$R_4:\ 3x-y=1$ $\quad S_4:\ U=1$

Constant → Always rectangle



Case 1:

$$U=3x-y \qquad\qquad U=3x-y$$
$$V=x+y \qquad\qquad -3V=-3x-3y$$
$$U+V=4x \qquad\qquad U-3V=-4y$$
$$x=\frac{U+V}{4} \qquad\qquad y=\frac{3V-U}{4}$$

$$\frac{4^{1}}{16_{4}}$$

Jacobian $\begin{vmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} \end{vmatrix} = \frac{3}{16} + \frac{1}{16} = \frac{1}{4}$

Jacobian

$$= \int_0^1 \int_0^1 U^{\frac{3}{2}} V^5 \left|\frac{1}{4}\right| dU\, dV$$

↓ DIY

Ans $= \left(\frac{1}{4}\right)\left(\frac{1}{15}\right) = \frac{1}{60}$ #

16. Evaluate the line integral $\int_C (xz + 2y)\, dS$, where C is the line segment from $(0, 1, 0)$ to $(1, 0, 2)$.

$\int F \cdot dr$

$\nabla f(x,y,z)$ is continuous and not equal to 0 hence we can apply line Integral.

Segment $= (1-t)(0,1,0) + t(1,0,2) \in t\,(0,1)$

$= (0, 1-t, 0) + (t, 0, 2t)$

$r(t) = \langle t, 1-t, 2t \rangle$

$\|r'(t)\| = \langle 1, -1, 2 \rangle = \sqrt{1+1+4} = \sqrt{6}$

$= \int f(r(t)) \, \|r'(t)\|$

$= \int_0^1 (2t^2 + 2 - 2t)\sqrt{6}\, dt$

$= \sqrt{6}\left[\frac{2t^3}{3} + 2t - t^2 \Big|_0^1\right]$

$= \sqrt{6}\left[\frac{2}{3} + 2 - 1\right] = \sqrt{6}\left[\frac{8}{3} - 1\right] = \sqrt{6}\left[\frac{5}{3}\right]$

$= \frac{\sqrt{6}(5)}{3}$ #

17. Let $F(x,y) = (xy^2 + 2y)i + (x^2y + 2x + 2)j$ be a vector field.

(a) Show F is conservative.

(b) Find f such that $\nabla f = F$.

(c) Evaluate $\int_C F\,dr$ where C is defined by $r(t) = \langle e^t, 1+t \rangle$, $0 \le t \le 1$.

(d) Evaluate $\int_C F\,dr$ where C is a closed curve $r(t) = \langle 2\sin(t), \cos(t) \rangle$, $0 \le t \le 2\pi$.

(a) F is conservative if $F = \nabla f$; as

assume F is open and simply connected

or $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$

Let $F(P,Q) = (\underbrace{xy^2 + 2y}_{P}, \overbrace{x^2y + 2x + 2}^{Q})$

$= \quad 2xy + 2 = 2xy + 2$ //

Prove Conservative

(b). Find Potential function $f \rightarrow F = \nabla f$

$\int_x f_x = \frac{x^2y^2}{2} + 2yx + g(y)$

$\int_y f_y = \frac{x^2y^2}{2} + 2xy + 2y + g_y(y)$

Hence $f(x,y) = \frac{x^2y^2}{2} + 2xy + 2y + K$ //

(c.) as $f(x,y)$ is continuous over the region $0 \le t \le 1$

and $f'(r(t)) \neq 0$

Hence we can apply Fundamental theorem of Line integral.

$\downarrow$

$$\int_C F \cdot dr = f(r(b)) - f(r(a)) \Big|_0^1$$

$$f(e^t, 1+t) = \frac{(e^t)^2(1+t)^2}{2} + 2(e^t)(1+t) + 2(1+t) \Big|_0^1$$

$$= \frac{4e^2}{2} + 2e^1(1+1) + 2(1+1)$$

$$= (2e^2 + 4e + 4) - \left(\frac{(1)^2(1)^2}{2} + 2(1)(1) + 2 \right)$$

$$= (2e^2 + 4e + 4) - \left(\frac{1}{2} + 2 + 2 \right)$$

$$= 2e^2 + 4e + 4 - \frac{9}{2} = 2e^2 + 4e - \frac{1}{2}$$

d.) (d) Evaluate $\int_C F\,dr$ where C is a closed curve $r(t) = \langle 2\sin(t), \cos(t) \rangle$, $0 \le t \le 2\pi$.

As it is a conservative vector field

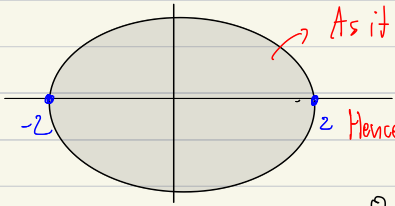
Hence $\int F \cdot dr = 0$

integrate a conservative vector field on a close curve will be equal to zero.

18. Use Green's Theorem to evaluate $\int_C -y^3\,dx + x^3\,dy$ where C is a circle given by $r(t) = \langle 2\cos t, 2\sin t\rangle$, $0 \le t \le 2\pi$.

$P\,dx + Q\,dy$

Circle with radius of 2.



As it is a simple close curve. piecewise smooth, and positive oriented. Hence we can apply green theorem.

$0 \le r \le 2$, $\theta \Rightarrow 0 \le \theta \le 2\pi$

$$= \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

Let $f(P,Q) = (-y^3, x^3)$

$$\frac{\partial Q}{\partial x} = 3x^2, \quad \frac{\partial P}{\partial y} = -3y^2$$

$$= \int_0^{2\pi}\int_0^{2} (3x^2 + 3y^2)\, r\, dr\, d\theta$$

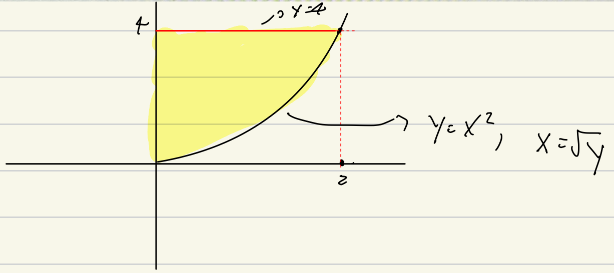
$$= \int_0^{2\pi}\int_0^{2} 3\underbrace{(x^2+y^2)}_{r^2}\, r\, dr\, d\theta$$

$$= 3\left[\int_0^{2\pi}\int_0^{2} r^3\, dr\, d\theta\right] \text{ both constant } a, b, c, d$$

$$= 3\left[2\pi\left[\frac{r^4}{4}\Big|_0^2\right]\right] = 3\left[2\pi\left(\frac{2^4}{4}\right)\right] = 3\left[2\pi\left(4\right)\right]$$

$$= 3\left[8\pi\right] = 24\pi$$

19. Use Green's Theorem to evaluate $\int_C (e^x+y^2)\,dx+(e^y+x^2)\,dy$ where C is the positively oriented boundary of the region in the first quadrant bounded by $y=x^2$ and $y=4$.



Simple close curved, Positive oriented, piecewise smooth such that we can apply green theorem.

$$F(P,Q) = (e^x+y^2,\ e^y+x^2)$$

$$= \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

$$\frac{\partial Q}{\partial x} = 2x, \quad \frac{\partial P}{\partial y} = 2y$$

$$= \int_0^4 \int_0^{\sqrt{y}} 2x - 2y \, dx\, dy$$

$$= \int_0^4 x^2 - 2yx \Big|_0^{\sqrt{y}} dy \qquad \frac{1}{2}+1$$

$$= \int_0^4 y - 2y\sqrt{y}\, dy = \int_0^4 y - 2y^{\frac{3}{2}}\, dy = \frac{y^2}{2} - \frac{4}{5}y^{\frac{5}{2}} \Big|_0^4$$

$$= 8 - \frac{4(4)^{\frac{5}{2}}}{5}$$

$$= 8 - \frac{4\left((4)^{\frac{1}{2}}\right)^5}{5}$$

$$= 8 - \frac{4(2)^5}{5} = 8 - \frac{4(32)}{5} = 8 - \frac{128}{5}$$

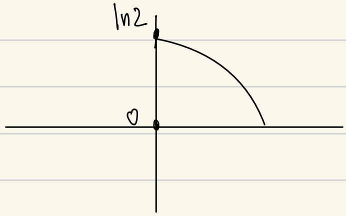
Final Exam 3 Math with Professor V

Change the Cartesian integral to an equivalent polar integral, and then evaluate. You must include a labeled graph of the integration region (2D).

2) $\displaystyle\int_0^{\ln 2}\int_0^{\sqrt{(\ln 2)^2 - y^2}} e^{\sqrt{x^2+y^2}}\,dx\,dy$

2)

$$x = \sqrt{(\ln 2)^2 - y^2}$$
$$x^2 + y^2 = (\ln 2)^2$$
$$r^2 = (\ln 2)^2$$
$$r = \ln 2$$



$0 \le r \le \ln 2$

$0 \le \theta \le \frac{\pi}{2}$

!! อย่าสะเพร่า

$$= \int_0^{\frac{\pi}{2}}\int_0^{\ln 2} e^r\, r\, dr\, d\theta$$

$$\frac{\pi}{2}\left[\int_0^{\ln 2} re^r\, dr\right]$$

$$= \frac{\pi}{2}\left[\left. re^r - e^r \right|_0^{\ln 2}\right]$$

$$= \frac{\pi}{2}\left[\left(\ln 2\, e^{\ln 2} - e^{\ln 2}\right) - (0 - 1)\right]$$

$$= \frac{\pi}{2}\left[(2\ln(2) - 2) + 1\right] = \frac{\pi}{2}\left(2\ln(2) - 1\right)$$

#

3) $\int_C \left[\dfrac{x^2+y^2}{z^2}\right] ds$, C is the curve $\mathbf{r}(t) = (10 \sin \frac{1}{2}t)\mathbf{i} + (10 \cos \frac{1}{2}t)\mathbf{j} + 12t\mathbf{k}$, $2 \le t \le 4$

$$= \int_2^4 F(r(t))\,\|r'(t)\|\,dt$$

ไม่ต้องเอา $\frac{1}{2}$ แยกออกมา

as $\sin^2(\theta) + \cos^2(\theta) = 1$

θ จะเป็นอะไรก็ได้

$$F(r(t)) = \frac{(10\sin\frac{1}{2}t)^2 + (10\cos(\frac{1}{2}t))^2}{144t^2}$$

$$= \frac{100\sin^2(\frac{1}{2}t) + 100\cos^2(\frac{1}{2}t)}{144t^2} = \frac{100}{144t^2}$$

$$\|r'(t)\| = \langle 5\cos(\tfrac{1}{2}t), -5\sin(\tfrac{1}{2}t), 12\rangle$$

$$= \sqrt{(5\cos(\tfrac{1}{2}t))^2 + (-5\sin(\tfrac{1}{2}t))^2 + (12)^2}$$

$$= \sqrt{25(\underbrace{\cos^2(\tfrac{1}{2}t) + \sin^2(\tfrac{1}{2}t)}_{1}) + 144}$$

$$= \sqrt{169} = 13$$

$$= \int_2^4 f(r(t))\,\|r'(t)\| = \int_2^4 \frac{100}{144t^2}(13)$$

$$-\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$$

$$= \frac{100}{144}(13)\left[-\frac{1}{t}\Big|_2^4\right] = \frac{\cancel{100}^{25}}{144}(13)\cdot\frac{1}{\cancel{4}}$$

$$= \frac{325}{144}$$

Find the work done by F over the curve in the direction of increasing t. (Hint: you do not actually need to compute the line integral)

4) $\mathbf{F} = xe^{4x^2}\mathbf{i} + e^{2y}\mathbf{j} + e^{8z}\mathbf{k}$; the path is $C_1 \cup C_2$ where C_1 is the straight line from (0, 0, 0) to (1, 1, 0) and C_2 is the straight line from (1, 1, 0) to (1, 1, 1)

Note for a close curve: An integral for a conservative vector field is equal to zero.

Check if F is conservative such that $\nabla \times F = 0$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xe^{4x^2} & e^{2y} & e^{8z} \end{vmatrix} = \langle 0-0, 0, 0 \rangle$$

Hence F is conservative

If F is conservative then $F = \nabla f$

let $F(P, Q, R) = (xe^{4x^2}, e^{2y}, e^{8z})$

$u = 4x^2$

Find Potential Function = $\int_x F_x = \frac{1}{8}e^{4x^2}$, $\int_y F_y = \frac{1}{2}e^{2y}$, $\int_z F_z = \frac{1}{8}e^{8z}$

$$f(x,y,z) = \frac{1}{8}e^{4x^2} + \frac{1}{2}e^{2y} + \frac{1}{8}e^{8z} + K$$

As C_1 and C_2 is continuous and $f(x,y,z)$ is a polynomial hence continuous also in the set of C_1 and C_2. Thus, we can apply Fundamental of Line Integral.

$$\int_{C_1} f(x,y,z) \int_{2} f(x,y,z) = f(r(b)) - f(r(a)) \Big|_a^b$$

$(1,1,1) \longrightarrow (1,1,1)$

$$= \frac{1}{8}e^{4x^2} + \frac{1}{2}e^{2y} + \frac{1}{8}e^{8z} \Big|_{(0,0,0)}^{(1,1,1)}$$

: Note:

Line integral is path independent, so we don't care about the path. We only care abt start and end p.n.t. In this Question as $C_1 \cup C_2$ then it is connected together and

$$\left[\left(\frac{1}{8}e^4+\frac{1}{2}e^2+\frac{1}{8}e^8\right)-\left(\frac{1}{8}+\frac{1}{2}+\frac{1}{8}\right)\right].$$

can use the end point for C_2 and start point to C_1. This way we don't need to separate to $\int_{C_1} dt \int_{C_2} dt$

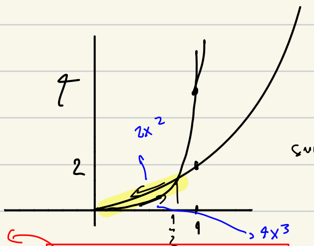
$$\left[\left(\frac{1}{8}e^4+\frac{1}{2}e^2+\frac{1}{8}e^8\right)-\left(\frac{6}{8}\right)\right]$$

$$=\frac{1}{8}e^4+\frac{1}{2}e^2+\frac{1}{8}e^8-\frac{6}{8}$$

Using Green's Theorem, calculate the area of the indicated region. You must include a clearly labeled graph of the region.

5) The area bounded above by $y = 2x^2$ and below by $y = 4x^3$

cheat sheet => use area for green theorem



Closed simple curve, piece wise smooth, positive orientation such that we can apply green theorem.

green theorem

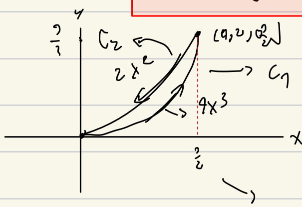
$$\boxed{\text{Area} = \oint_C x\,dy}$$

intersection: $2x^2 = 4x^3$

$4x^3 - 2x^2 = 0$

$2x^2(2x-1) = 0$

$x = \frac{1}{2},\ x = 0$



→ Parameter cannot decrease $\int_{1/2}^{0} \neq$!! NO

C_1: $x = t$
$y = 4t^3$
$0 \le t \le \frac{1}{2}$
$dy = 12t^2$

C_2. $x = \frac{1}{2} - t$
$y = 2(\frac{1}{2}-t)^2$
$0 \le t \le \frac{1}{2}$
$dy = -4(\frac{1}{2}-t)$

$$A = \oint_C x\,dy = \underbrace{\int_0^{1/2} t \cdot 12t^2}_{C_1} + \int_0^{1/2} \left(\frac{1}{2}-t\right)(-4)\left(\frac{1}{2}-t\right) dt$$

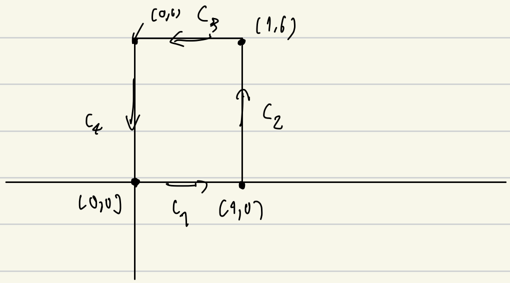
$$= \int_0^{\frac{1}{2}} 12t^3 - 4\left(\frac{1}{2} - t\right)^2 \, dt$$

$$= \left[3t^4 + \frac{4}{3}\left(\frac{1}{2} - t\right)^3\right]_0^{\frac{1}{2}}$$

$$OIY = \boxed{\frac{1}{48}}$$

Using Green's Theorem, compute the counterclockwise circulation of F around the closed curve C. You must include a clearly labeled graph of the region enclosed by C.

6) **F** = $(x^2 + y^2)$**i** + $(x - y)$**j**; C is the rectangle with vertices at (0, 0), (1, 0), (1, 6), and (0, 6)



Simple close curve, positive orientation, piece wise smooth enough such that we can apply green theorem.

Green theorem $\displaystyle\int_C F \cdot dr \, dA = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA$

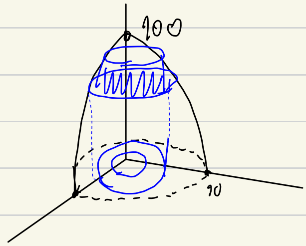
let $F(P, Q) = (x^2 + y^2, x - y)$ = $\dfrac{\partial Q}{\partial x} = 1$, $\dfrac{\partial P}{\partial y} = 2y$

$$= \int_0^6 \int_0^1 1 - 2y \, dx \, dy$$

$$= \int_0^6 1 - 2y \, dy = y - y^2 \Big|_0^6 = 6 - 36 = -30$$

Find the surface area of the surface S. You must include a clearly labeled graph of the integration region (2D).

7) S is the portion of the paraboloid $z = 100 - x^2 - y^2$ that lies above the ring $25 \le x^2 + y^2 \le 49$ in the xy-plane.



Surface Area $= \iint_S dS \to \sqrt{f_x^2 + f_y^2 + 1}$

$dS = \sqrt{z_x^2 + z_y^2 + 1}$

$f_x = -2x, \quad f_y = -2y$

$= \sqrt{(-2x)^2 + (-2y)^2 + 1} = \sqrt{4x^2 + 4y^2 + 1} = \sqrt{4(x^2+y^2)+1}$

$\sqrt{4r^2+1}$

$25 \le x^2 + y^2 \le 49$

$5 \le r \le 7$

$$= \int_0^{2\pi} \int_5^7 \sqrt{4r^2+1}\; r\, dr\, d\theta$$

$$= 2\pi \int_5^7 r\sqrt{4r^2+1}\, dr$$

$\overset{3}{49} \times 4$

196

$u = 4r^2 + 1 \qquad du = 8r\, dr$

$dr = \frac{du}{8r}$

$$= \frac{\pi}{4} \int_{101}^{197} \sqrt{u}\, du$$

$$= \frac{\pi}{\cancel{4}\,2} \left[\frac{\cancel{2} u^{\frac{3}{2}}}{3} \Big|_{101}^{197} \right]$$

$$= \frac{\pi}{6} \left[197\sqrt{197} - 101\sqrt{101} \right]$$

Find the flux of the curl of field F through the shell S. You must include a clearly labeled graph of the surface (3D) and the integration region (2D).

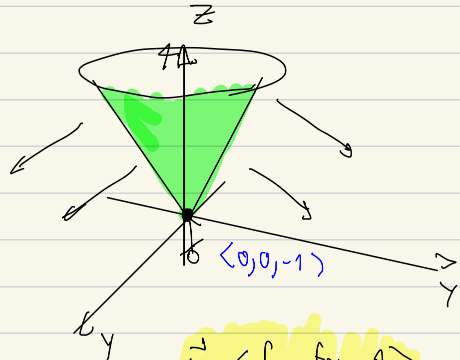
8) $\mathbf{F} = -3z\mathbf{i} + 3x\mathbf{j} + 9y\mathbf{k}$; S is the portion of the cone $z = 5\sqrt{x^2 + y^2}$ below the plane $z = 4$

$$W = \iint_S \text{curl}\,\vec{F}\cdot\vec{n}$$

$$W = \iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,ds \qquad \vec{F} = \langle -3z, 3x, 9y \rangle$$

Stoke

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -3z & 3x & 9y \end{vmatrix} = (9-0, -3, 3)$$



$z = 5\sqrt{x^2+y^2}$

$\frac{4}{5} = r$

$0 \le r \le \frac{4}{5}$

$\vec{n} = \langle f_x, f_y, -1 \rangle$

Use this.

$\vec{n} = \langle -f_x, -f_y, 1 \rangle$

$$\vec{n} = \left\langle \frac{5x}{\sqrt{x^2+y^2}}, \frac{5y}{\sqrt{x^2+y^2}}, -1 \right\rangle$$

$\text{Curl}\,\vec{F}\cdot N$

$$= \frac{45x}{\sqrt{x^2+y^2}} - \frac{15y}{\sqrt{x^2+y^2}} - 3$$

$$= \int_0^{2\pi} \int_0^{4/5} \left(\frac{4\sqrt{5}\, r\cos\theta}{r} - \frac{15 r\sin\theta}{r} - 3 \right) r \, dr \, d\theta$$

$$\downarrow$$

$$DI_y = -6\pi$$

[2] (8 points) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by:

$$f(x, y, z) = 3z + e^{x^2 - y^2}$$

Let C be the set of the heads of unit vectors v in $\mathbb{R}^3$ such that f increases at $1/3$ of its maximum rate of change in the direction v starting from $(0, 0, 1)$. Find the equation(s) which determine(s) the set C. (Hint : C is a circle in $\mathbb{R}^3$.)

Maximum Rate of Change => Directional Derivative.

Math 212 Final Exam.

Optimization

[3] (8 points) Find the absolute minimum and maximum for the function $f(x, y) = x + 2y^2 + 1$ on the unit disk $D = \{(x, y) | x^2 + y^2 \le 1\}$.

Constraint

as $f(x,y)$ is a polynomial, Hence it is continuous and we can find its first Partial derivative.

$\therefore$ As $g(y)$ $\{x^2+y^2 \le 1\}$ is close and bounded then we can apply the method of Lagrange Multiplier.

$f(x,y) = x+2y^2+1 \rightarrow$ optimize , $g(x,y)$ $x^2+y^2 \le 1$, constraint

Close and bounded and hence can reach extreme min/max value.

Check if $\nabla g(x,y)$ vanish over f $\quad \nabla g = \langle 2x, 2y \rangle \neq 0$

Prove

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$
$$\langle 1, 4y \rangle = \lambda \langle 2x, 2y \rangle$$

$$2x\lambda = 1 \qquad \lambda = \frac{1}{2x}$$
$$2y\lambda = 4y$$
$$x^2+y^2 = 1$$

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$$2y\lambda - 4y = 0$$
$$2y(\lambda - 2) = 0$$
$$\lambda = 2, \; y = 0$$

When $\lambda = 2 \rightarrow$ $2 = \frac{1}{2x}$

$$4x = 1$$
$$x = \frac{1}{4}$$

when $y = 0$ $\quad x^2 = 1$

$$x = \pm 1$$

P.n.t 1 = $(1,0)$, $(-1,0)$

when $x = \frac{1}{4} \rightarrow$

$$\frac{1}{16} + y^2 = 1$$

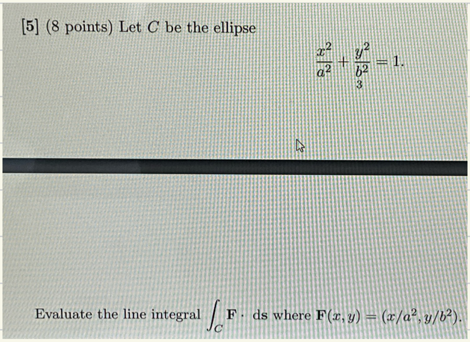
$$y^2 = 1 - \frac{1}{16} \rightarrow y^2 = \frac{15}{16} \quad y = \pm \frac{\sqrt{15}}{4}$$

Hence P.n.t 2 = $\left(\frac{1}{4}, \frac{\sqrt{15}}{4}\right)$ and $\left(\frac{1}{4}, -\frac{\sqrt{15}}{4}\right)$

Check p.n.t. $f(x,y) = x + 2y^2 + 1$

$$f(1,0) = 1 + 0 + 1 = 2$$
$$f(-1,0) = -1 + 0 + 1 = 0$$
$$f\left(\frac{1}{4}, \frac{\sqrt{15}}{4}\right) = \frac{1}{4} + 2\left(\frac{15}{16}\right) + 1 = \frac{1}{4} + \frac{15}{8} + 1 = \frac{17}{8} + 1 = \frac{25}{8} =$$
$$f\left(\frac{1}{4}, -\frac{\sqrt{15}}{4}\right) = \rightarrow \text{same} = \frac{25}{8}$$

Hence abs max at p.n.t $\left(\frac{1}{4}, \frac{\sqrt{15}}{4}\right), \left(\frac{1}{4}, -\frac{\sqrt{15}}{4}\right)$ Min at $(-1, 0)$ #



[5] (8 points) Let C be the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

3

Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{s}$ where $\mathbf{F}(x,y) = (x/a^2, y/b^2)$.

Paramatize r(t)

$\downarrow$

Let $r(t) = \begin{array}{l} x = a\cos\theta \\ b = b\sin\theta \end{array}$

$$r'(t) = \langle -a\sin\theta, b\cos\theta \rangle$$

$$\int F \cdot dr = \int F(r(t)) \cdot r'(t)\, dt$$

$$= \int \left(\frac{\cancel{a}\cos\theta}{\cancel{a^2}}, \frac{\cancel{b}\sin\theta}{\cancel{b^2}} \right) \cdot \left(-a\sin\theta, b\cos\theta \right)$$

$$= \int_0^{2\pi} -\sin\theta\cos\theta + \sin\theta\cos\theta$$

$$= 0$$

Hence the line integral = 0.

[6] (8 points) Evaluate the line integral $\int e^{x^2}\, dx - xy\, dy + y^2\, dz$, where $\mathbf{c}(t) = (1, t, t^2)$, $0 \le t \le 1$.

$c(t) = (1, t, t^2)$

$dx = 0\, dt \quad dy = 1\, dt \quad dz = 2t\, dt$

$$= \int_0^1 e^1(0) - (1)(t)(1) + t^2(2t)\, dt$$

$$= \int_0^1 -t + 2t^3\, dt$$

$$= -\frac{t^2}{2} + \frac{t^4}{2}\Big|_0^1 = -\frac{1}{2} + \frac{1}{2} = 0$$

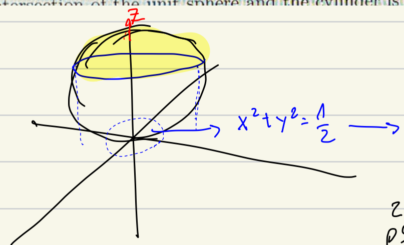
$r^2 = \frac{1}{2}$

$r = \sqrt{\frac{1}{2}} \qquad r = \pm\frac{1}{\sqrt{2}}$

$0 \le r \le \frac{1}{\sqrt{2}}$

[7] (8 points) Find the area of the portion of the unit sphere inside the cylinder $x^2 + y^2 = \frac{1}{2}$ and $z > 0$.

Solution. The intersection of the unit sphere and the cylinder is a circle, ~~and~~ the angle between



$\rho^2 \sin^2\phi \sin^2\theta + \rho^2 \sin^2\phi \cos^2\theta = \frac{1}{2}$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \sin\phi\, d\phi\, d\theta = \left[-\cos\phi\right]_0^{\frac{\pi}{4}} \cdot 2\pi \qquad \rho^2\sin^2\phi = \frac{1}{2}$$

$= (2-\sqrt{2})\pi$

Ucla Final Exam.

(a) [7pts.] Let C be the half of the unit circle $x^2 + z^2 = 1$, $y = 0$, with $z \leq 0$. Orient C in the clockwise direction when viewed from the negative y-axis. Let $\mathbf{F}(x, y, z) = (y^4, z^3, 1 - x^2)$. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

let parametize $r(t) \Rightarrow$ let $r(t) = x = \cos\theta \quad z = \sin\theta$

$y = 0$

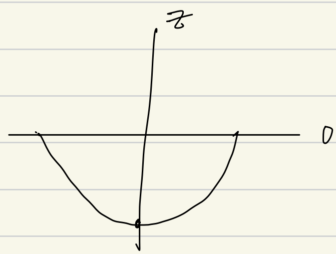
$\int F \cdot dr = \int F(r(t)) \cdot r'(t)\, dt \qquad r'(t) = (-\sin\theta, 0, \cos\theta)$

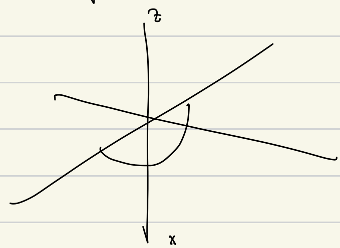
$= \int (0, \sin^3\theta, 1 - \cos^2\theta) \cdot (-\sin\theta, 0, \cos\theta)$

$= 0 + 0 + \cos - \cos^3\theta$

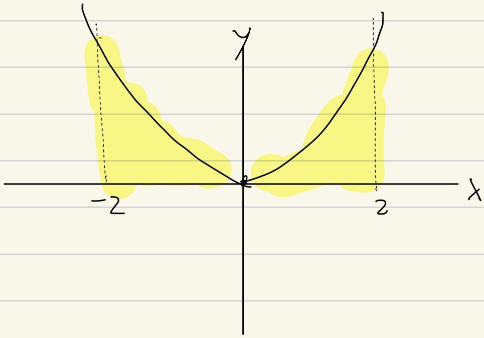
$= \int_{\pi}^{2\pi} \cos\theta - \cos^3\theta$

odd function $= 0$





) [7pts.] Let C be the curve in the xy-plane described by $z = 0$, $y = x^2$, $-2 \le x \le 2$. Let $f(x, y, z) = 12(x + \sqrt{y} + z^3)$. Calculate $\int_C f \, ds$.



$\int_C f \, ds = \int f(r(t)) \, \|r'(t)\|$

$t \in (0,$

let $r(t) = \langle t, t^2, 0 \rangle$, $\|r'(t)\| = \sqrt{(1)^2 + (2t)^2 + 0^2}$

$= \sqrt{1 + 4t^2}$

$= \int_{-2}^{2} 12(t + \sqrt{t^2})(\sqrt{1+4t^2}) \, dt$

Seperate when $t < 0 = \int_{-2}^{0} 12 \underbrace{(t - t)}_{=0} \sqrt{1+4t^2} \, dt$ +

$\int_{0}^{2} 12(t+t) \sqrt{1+4t^2} \, dt$

$= \int_0^2 24t \sqrt{1+4t^2}$ = DIY = $2(17^{\frac{3}{2}} - 1)$

Problem 2.

Let S be the part of the cone $x^2 + y^2 = 4z^2$ between the planes $z = 2$ and $z = 5$, and in the second xy-quadrant $x \le 0$, $y \ge 0$.

(a) [3pts.] Which two of the following vectors are tangent vectors to S at $(-3, 4, \frac{5}{2})$?

show → surface Integral.

b.) $\iint f\,dS = 39\sqrt{5}(\pi - 4)$ $= \iint f(r(u), r(v))\,\|r'(u) \times r'(v)\|\,du\,dv$

$x^2 + y^2 = 4z^2$

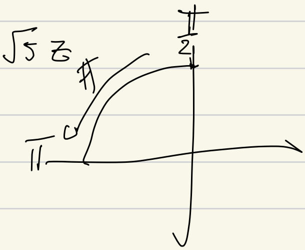
let $x = 2z\cos\theta$, $y = 2z\sin\theta$, $z = 2$ to 5

$r'(v)_\theta = (-2z\sin\theta,\ 2z\cos\theta,\ 0)$

$r'(v)_z = (2\cos\theta,\ 2\sin\theta,\ 1)$

$\|r'(u) \times r'(v)\| = OTY = \sqrt{4z^2(\cos^2\theta + \sin^2\theta) + 16z^2}$

$= \sqrt{20z^2} = 2\sqrt{5}z$ #



$dS = 2\sqrt{5}z\,d\theta\,dz$

$f(r(u), r(v))$ $\|r'(u) \times r'(u)\|$

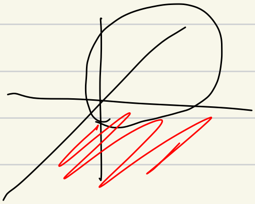
$$= \int_{\pi/2}^{\pi} \int_{2}^{5} (2z\cos\theta + z)(2\sqrt{5}z)\,dz\,d\theta$$

$OTY = 39\sqrt{5}(\pi - 4)$ #

Problem 3. *8pts.*

Let $\mathcal{E}$ be the region enclosed by the surfaces $x^2+y^2+z^2=8$ and $x^2+y^2=2z$ in the octant $x, y, z \geq 0$.

Let $f(x,y,z)=3z$. Calculate $\iiint_{\mathcal{E}} f\, dV$.



Can also use spherical but harder.

$z \geq 0$

when $z=0$

$r^2=8$

$r=\sqrt{8} \rightarrow 0 \leq r \leq \sqrt{8}$

$$x^2+y^2+z^2=8 \qquad\qquad x^2+y^2=2z$$

$$z^2=8-x^2-y^2 \qquad\qquad z=\frac{x^2+y^2}{2}$$

$$z=\sqrt{8-r^2} \qquad\qquad z=\frac{r^2}{2}$$

จับ z เท่ากัน เพื่อหา r

$$\sqrt{8-r^2}=\frac{r^2}{2} \Rightarrow \left(2\sqrt{8-r^2}\right)^2=\left(r^2\right)^2$$

$$4(8-r^2)=r^4$$

$$32-4r^2=r^4$$

$$r^4+4r^2-32=0$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{2}\int_{\frac{r^2}{2}}^{\sqrt{8-r^2}} 3z\, dz\, r\, dr\, d\theta$$

$$r^2=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-4\pm\sqrt{16-4(1)(-32)}}{2(1)}$$

$$=\frac{-4\pm 12}{2}=\frac{8}{2}=4 \text{ or } -8$$

use $r^2=4$

DIY $=7\pi$

$r = 2$

Problem 4.

In each of the parts (a)-(d), you should justify your answer fully.

(a) [4pts.] Is $\mathbf{F}(x,y,z) = \frac{(x,y,0)}{x^2+y^2}$ conservative on $\mathbb{R}^3 - (z\text{-axis})$?

(b) [4pts.] Is $\mathbf{F}(x,y,z) = \frac{(-y,x,0)}{x^2+y^2}$ conservative on $\mathbb{R}^3 - (z\text{-axis})$?

(c) [4pts.] Does $\mathbf{F}(x,y,z) = (x,y,z)$ have a vector potential on $\mathbb{R}^3 - \{(0,0,0)\}$?

(d) [4pts.] Does $\mathbf{F}(x,y,z) = (0,0,1)$ have a vector potential on $\mathbb{R}^3 - \{(0,0,0)\}$?

(e) [2pts.] Write down a domain on which there exist well-defined differentiable vector fields $\mathbf{F}$ and $\mathbf{G}$ such that $\nabla \times \mathbf{F} = \mathbf{0}$, $\nabla \cdot \mathbf{G} = 0$, $\mathbf{F}$ is not conservative, and $\mathbf{G}$ does not have a vector potential.

$\frac{x}{x^2+y^2}\,dx$

$= u = x^2+y^2$

$du = 2x\,dx$

$dx = \frac{du}{2\cancel{x}}$

(a). Conservat $F = \nabla f$

as $F(x,y,z) = \frac{(x,y,0)}{x^2+y^2}$

We find Potential Function first

$\int_x f_x = \frac{1}{2}\ln(x^2+y^2)$, $\int_y f_y = \frac{1}{2}\ln(x^2+y^2)$, $\int_z f_z = 0$

Potential function $= f(x,y,z) = \frac{1}{2}\ln(x^2+y^2) + g(y)$

Chek if $F = \nabla f \rightarrow \overset{?}{=} \left(\frac{x}{\ln(x^2+y^2)}, \frac{y}{\ln(x^2+y^2)}, 0 \right)$

same hence Conservative.

b.) $\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix} = ((0-0), 0, 0,$ Non zero $)$

Hence by Curl it is not conservative

c.) $\nabla \cdot F$ } Divergence $= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3 \neq 0$

Hence the divergence, its not Conservative.

d. Yes use $\nabla \times A = F$ (Curl)

e. $[1, 0, 0]$