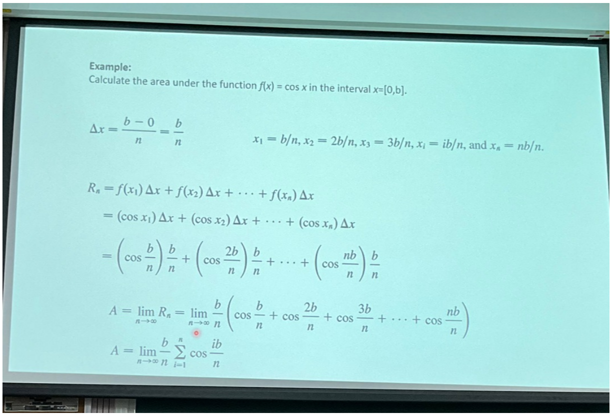
Ex1. Calculate the area under $f(x) = \cos x$ interval $x = [0, b]$



Example:

Calculate the area under the function $f(x) = \cos x$ in the interval $x=[0,b]$.

$$\Delta x = \frac{b-0}{n} = \frac{b}{n} \qquad x_1 = b/n, x_2 = 2b/n, x_3 = 3b/n, x_i = ib/n, \text{ and } x_n = nb/n.$$

$$R_n = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x$$

$$= (\cos x_1)\,\Delta x + (\cos x_2)\,\Delta x + \cdots + (\cos x_n)\,\Delta x$$

$$= \left(\cos \frac{b}{n}\right)\frac{b}{n} + \left(\cos \frac{2b}{n}\right)\frac{b}{n} + \cdots + \left(\cos \frac{nb}{n}\right)\frac{b}{n}$$

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{b}{n}\left(\cos \frac{b}{n} + \cos \frac{2b}{n} + \cos \frac{3b}{n} + \cdots + \cos \frac{nb}{n}\right)$$

$$A = \lim_{n\to\infty} \frac{b}{n} \sum_{i=1}^{n} \cos \frac{ib}{n}$$

* Riemann sum

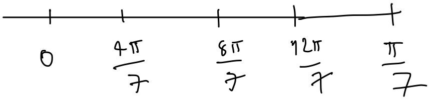
Program to calculate area under curve for python x^2 with $n=\infty$

$\cos(x)$ for $b = \frac{\pi}{7}$, $n = 4$ → HW 3

$$\text{Area} = \sum_{i=1}^{4} \Delta x \, f(x_i)$$

$$\Delta x = \frac{\frac{\pi}{7} - 0}{4} = \frac{4\pi}{7}$$

4



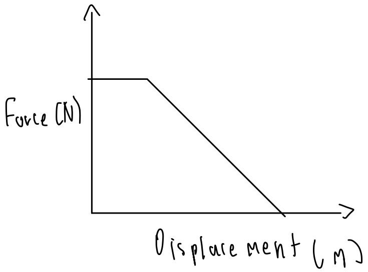
$$A_L = \frac{4\pi}{7}\left[f(0) + f\left(\frac{4\pi}{7}\right) + f\left(\frac{8\pi}{7}\right) + f\left(\frac{12\pi}{7}\right)\right]$$

$$= \frac{4\pi}{7}\left[1 + (-0.222) + (-0.9) + (0.623)\right]$$

$$= 0.8993$$

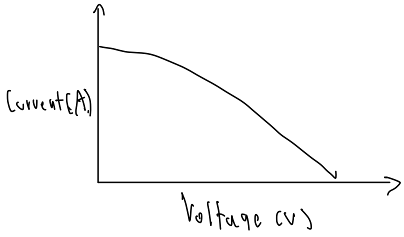
HW # 1

1. force versus Displacement graph



= Work done = J or NM

2. Voltage versus Current graph

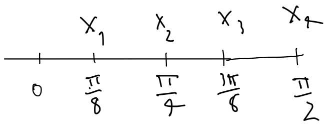


= Energy = W/s .

$\cos(x)$ for $b = \frac{\pi}{2}$, $n = 4$ → HW 3

$$\text{Area} = \sum_{i=1}^{4} \Delta x \, f(x_i)$$

$$\Delta x = \frac{\frac{\pi}{2} - 0}{4} = \frac{\pi}{8}$$



$$A_L = \frac{\pi}{8}\left[\frac{1}{2} f(0) + f\left(\frac{\pi}{8}\right) + f\left(\frac{\pi}{4}\right) + f\left(\frac{3\pi}{8}\right)\frac{1}{2}\right]$$

use mid point

$$= \frac{\pi}{16}\left[\cos(0) + 2\cos\left(\frac{\pi}{8}\right) + 2\cos\left(\frac{\pi}{4}\right) + 2\cos\left(\frac{3\pi}{8}\right) + \cos\left(\frac{\pi}{2}\right)\right]$$

$$= \frac{\pi}{16}\left[1 + 1.847 + \sqrt{2} + 0.765 + 0\right] \approx 0.987$$

#

Difference between integral and Anti derivative

integral — พื้นที่ใต้เส้นฟังชั่น

antiderivative — ฟังชั่นของพื้นที่ใต้เส้นฟังชั่น

Riemann Sums.

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x = \int_a^b f(x)\, dx$$

$$x_i = x_0 + i \Delta x$$

Sum rules

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$\downarrow$

$$i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

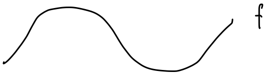
Darboux Integral

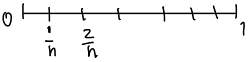
*

Supremum = least upperbound

Infremum = greatest lowerbound

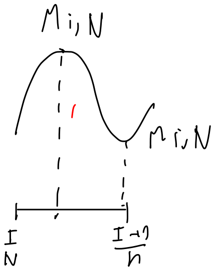
Given $f: [0,1] \longrightarrow R$





$1 = \frac{N}{N}$

$\downarrow$ Rectangle don't requried to have same size



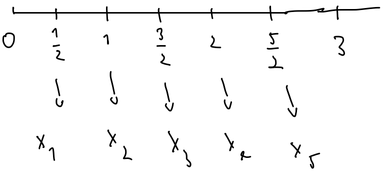
$$\rightarrow M_{i,n} = \sup \left\{ f(x) \middle| \, x \in \left[\frac{I}{N}, \frac{i+1}{n}\right] \right\}$$

$$m_{i,N} = \mathrm{INF}\{f(x)\}$$

DEF IF $U = L$ Then $\int_0^1 f(x)\, dx =: U = L$

Ex. $f(x) = x^2 - 6x$, in the range $[0, 3]$ for $n = 6$

$$\Delta x = \frac{3-0}{6} = \frac{1}{2}$$



Theorem

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $\Delta x = \frac{b-a}{n}$, $x_i = a + i\Delta x$

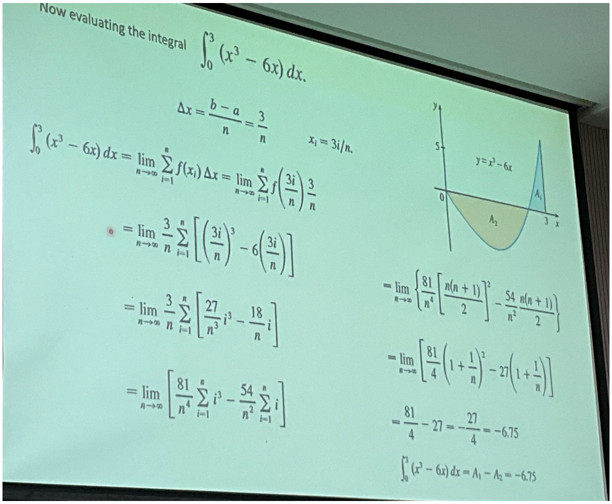
Right end point ↙ $R_6 = \sum_{i=1}^{6} f(x_i)\,\Delta x$

$= f(0.5)\Delta x + f(1.0)\Delta x + f(\frac{3}{2})\Delta x + f(2)\Delta x + f(\frac{5}{2})\Delta x$

$= -3.9375$ //

Evaluate integral $\int_0^3 (x^3 - 6x)\,dx$

$\Delta x = \frac{b-a}{n} = \frac{3}{n}$ $\quad x_i = \frac{3i}{n}$


Now evaluating the integral $\int_0^3 (x^3 - 6x)\,dx.$

$$\Delta x = \frac{b-a}{n} = \frac{3}{n} \qquad x_i = 3i/n.$$

$$\int_0^3 (x^3 - 6x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\frac{3}{n}$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right]$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\frac{27}{n^3} i^3 - \frac{18}{n} i\right]$$

$$= \lim_{n\to\infty} \left[\frac{81}{n^4} \sum_{i=1}^{n} i^3 - \frac{54}{n^2} \sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to\infty} \left\{\frac{81}{n^4}\left[\frac{n(n+1)}{2}\right]^2 - \frac{54}{n^2}\frac{n(n+1)}{2}\right\}$$

$$= \lim_{n\to\infty} \left[\frac{81}{4}\left(1 + \frac{1}{n}\right)^2 - 27\left(1 + \frac{1}{n}\right)\right]$$

$$= \frac{81}{4} - 27 = -\frac{27}{4} = -6.75$$

$$\int_0^3 (x^3 - 6x)\,dx = A_1 - A_2 = -6.75$$

Useful sum rules:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Midpoint Rule

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x = [f(\bar{x}_1) + \dots f(\bar{x}_n)]$$

$\Delta x = \frac{b-a}{n}$, $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$

↓

midpoint

Use the midpoint rule ex.

$\int_1^2 \frac{1}{x}\,dx \quad n=5 \quad \Delta$

$\Delta x = \frac{2-1}{5} = \frac{1}{5}$

Area $= \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x$

$x_1 = \frac{1}{5}$ $\quad x_2 = \frac{6}{5}$ $\quad x_3 = \frac{7}{5}$ $\quad x_4 = \frac{8}{5}$ $\quad x_5 = \frac{9}{5}$

1.1 $\quad$ 1.3 $\quad$ 1.5 $\quad$ 1.7 $\quad$ 1.9

$= \frac{1}{5}\left(\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}\right) \to$ Ans

Evaluate using midpoint rule $\int_0^3 (x^3 - 6x)\,dx$ $\qquad 0 + \frac{3i}{n}$

$\Delta x = \frac{3-0}{n} = \frac{3}{n}$ $\qquad \bar{x}_i = \frac{1}{2}($

$= \sum_{i=1}^{n}$

$x_1 = \frac{3}{n}$ $\quad x_2 = \frac{6}{n}$ $\quad x_3 = \frac{9}{n}$ $\quad x_4 = \frac{12}{n}$

$x_i = \frac{\frac{i6}{n}}{2}$ $\qquad x_n = \frac{n6}{n}$

$= \frac{i6}{2n}$

$= \sum_{i=1}^{n}\left(\frac{i6}{2n}\right)\left(\frac{3}{n}\right)$

$= \frac{3}{n}\sum_{i=1}^{n}\left(\frac{i6}{2n}\right)$

$= \lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\frac{3}{n}$

$= \left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)$

$= \frac{27i^3}{n^3} - \frac{18i}{n}$

Calculate $\int_0^1 (4+3x^2)\,dx$ using Definite integrals

$$\lim_{n\to\infty} \frac{3}{n}\left[\frac{27}{n^3}\sum_{i=1}^{n} i^3 - \frac{18}{n}\sum_{i=1}^{n} i\right]$$

$$\lim_{n\to\infty} \frac{81}{n^4}\left[\frac{n(n+1)}{2}\right]^2 - \frac{54}{n^2}\,\frac{n(n+1)}{2}$$

$$\lim_{n\to\infty} \frac{81}{4} - \frac{54}{2} = -\frac{27}{4}$$

Is the function integrale based on upper and lower Reimann sums.

Finals or Mid term

Properties of Definite integral

calculte infremum, suprem and see it converge or not.

The function $f(x) = \frac{1}{x}$ for $0 < x \leq 1$ and $f(x) = 0$ for $x = 0$

is integratable or not for interval $[0, 1]$.

*

For a function to be Riemann integrale, [the upper and lower sums must converge to the same number]

Fundamental Theorem of Calculus

If f is continuous on $[a, b]$ then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Ex. $\int_{-2}^{1} x^3 \, dx$

$$= \frac{x^4}{4} \Big|_{-2}^{1}$$

$$= \frac{(1)^4}{4} - \frac{(-2)^4}{4} = \frac{1}{4} - \left[\frac{16}{4}\right]$$

$$= \frac{1}{4} - 4 = -\frac{15}{4}$$

#

Find the area under curve $\cos(x)$ in $[0, b]$, where $0 \le b \le \frac{\pi}{2}$ #

$$\int_0^{\frac{\pi}{2}} \cos(x)\, dx$$

$$= \sin(x) \Big|_0^{b}$$

$$= \sin(b) - \sin(0) = 1 - 0 = 1$$

$$= \sin b$$

Indefinite Integral

+C

$$= \int 10x^4 - 2\sec^2 x \, dx$$

$$= \int 10x^4 - 2\int \sec^2 x \quad dx$$

$$= 10 \cdot \frac{x^5}{5} - 2\tan x$$

$$= 2x^5 - 2\tan x = 2(x^5 - \tan x) + C$$

#

$$= \int \frac{\cos\theta}{\sin^2\theta}\, d\theta$$

$u = \sin^2\theta \qquad du = 2\sin\theta \cdot \cos\theta\, d\theta$

$$d\theta = \frac{du}{2\sin\theta \cdot \cos\theta}$$

$$= \int \frac{\cancel{\cos}\theta}{u} \cdot \frac{du}{2\sin\theta\,\cancel{\cos\theta}}$$

$\sqrt{u} = \sin\theta$

$$= \int \frac{1}{u} \cdot \frac{du}{}$$

$$= \frac{1}{2}\int \frac{1}{u} \cdot \frac{du}{\sin\theta}$$

$$= \frac{1}{2}\int \frac{1}{u} \cdot \frac{d\theta}{\sqrt{u}}$$

$-\frac{3}{2} + 1$

$-\frac{1}{2}$

$$= \frac{1}{2}\int \frac{1}{u^{\frac{3}{2}}}$$

$-\frac{1}{2}$

$$= \frac{1}{2}\left[\frac{u^{-\frac{1}{2}}}{-\frac{1}{2}}\right]$$

$$= \frac{1}{2}\left[-\frac{2}{\sqrt{u}}\right] = -\frac{1}{\sin\theta} + C$$

$\cancel{\theta}$

$$\int_{0}^{12} (x - 12\sin x \quad dx$$

$$= \int x - 12 \int \sin x \quad dx$$

$$= \frac{x^2}{2} - 12 - \cos x$$

$$= \frac{x^2}{2} + 12 \cos x$$

$$= \left[\frac{(12)^2}{2} + 12\cos(12) \right] - \left[12 \right] \qquad = 72 + 12\cos(12) - 12$$

#

$$\int_{0}^{3} (x^3 - 6x) \quad dx$$

$$= \frac{x^4}{4} - 3x^2 \Big|_{0}^{3}$$

$$= \left[\frac{81}{4} - 27 \right] - \left[0 \right]$$

$$= \frac{81}{4} - 27$$

#

$$\int_{1}^{9} \frac{2t^2 + t^2\sqrt{t} - 1}{t^2} \, dt \qquad \frac{1}{t^2}$$

$$= 2t + \frac{2t^{\frac{3}{2}}}{3} - \left[\frac{t^{-1}}{-1}\right]$$

$$= 2t + \frac{2t^{\frac{3}{2}}}{3} + \frac{1}{t} \Bigg|_{1}^{9}$$

$$= \left[18 + \frac{54}{3} + \frac{1}{9}\right] - \left[2 + \frac{2}{3} + 1\right] =$$

$$36,11 - 3.66\bar{6} = 32.443$$

#

Integral

The integral of a rate of change is the net change

$$\int_a^b F'(x)\, dx = F(b) - F(a)$$

Ex: $V'(t)$: rate of the input flow at time t.

$V(t)$: amount of water and flow

$C'(t)$ = the reaction rate at time t

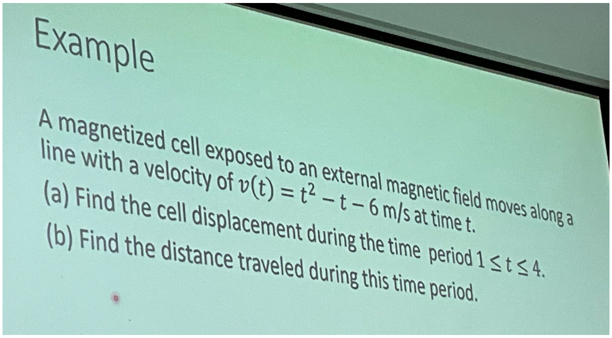
$C(t)$: the conkentration at time t.

$x(t)$ = position

$v(t) = x'(t)$: velocity

$$\text{displacement} = \int_{t_1}^{t_2} v(t)\, dt = A_1 - A_2 + A_3$$

$$\text{distance} = \int_{t_1}^{t_2} |v(t)|\, dt = A_1 + A_2 + A_3$$



Example

A magnetized cell exposed to an external magnetic field moves along a line with a velocity of $v(t) = t^2 - t - 6$ m/s at time t.

(a) Find the cell displacement during the time period $1 \le t \le 4$.

(b) Find the distance traveled during this time period.

a. $$\int_1^4 (t^2 - t - 6)\,dt$$

$$= \frac{t^3}{3} - \frac{t^2}{2} - 6t \Big|_1^4 = \left[\frac{64}{3} - \frac{16}{2} - 24\right] - \left[\frac{1}{3} - \frac{1}{2} - 6\right]$$

$$= -\frac{9}{2}$$

#

b. find distance

$v(t) = t^2 - t - 6 = (t-3)(t+2) \longrightarrow v(t) \le 0$ on $[1, 3]$

$v(t) \ge 0$ on $[3, 4]$

$$\longrightarrow \int_1^4 |v(t)|\,dt = \int_1^3 -v(t)\,dt + \int_3^4 v(t)\,dt$$

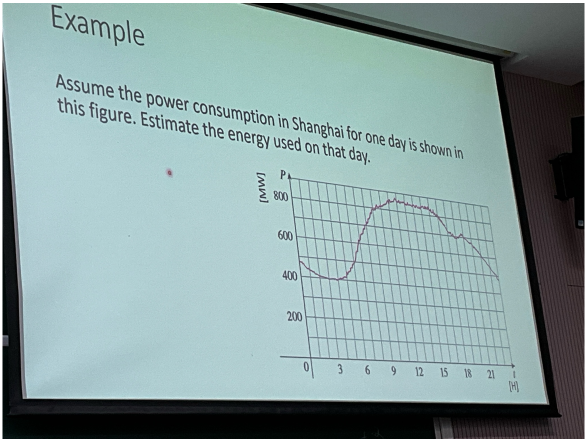
$$= \int_1^3 (-t^2 + t + 6)\,dt + \int_3^4 (t^2 - t - 6)\,dt$$

DIY

Ans $= \frac{61}{6} \approx 10.17$ m

unit of measurement for an integral.

$$\int_a^b f(x)\,dx = \text{Product of unit for } \Delta x \text{ and unit for } f(x).$$



Example

Assume the power consumption in Shanghai for one day is shown in this figure. Estimate the energy used on that day.

Sub Rule

ex.1. $\int 2x\sqrt{1+x^2}\ dx$

$u = 1 + x^2 \qquad du = 2x\ dx$

$dx = \frac{du}{2x}$

$= \int \cancel{2x}\sqrt{u} \cdot \frac{du}{\cancel{2x}}$

$= \int \sqrt{u}\ du$

$= \frac{2u^{\frac{3}{2}}}{3} = \frac{2(1+x^2)^{\frac{3}{2}}}{3} + C$

#

ex.2 $\int x^3 \cos(x^4+2)\ dx$

$u = x^4 + 2 \qquad du = 4x^3\ dx$

$dx = \frac{du}{4x^3}$

$= \int \cancel{x^3} \cos u \cdot \frac{du}{4\cancel{x^3}}$

$= \frac{1}{4}\int \cos u \cdot du$

$\therefore \frac{1}{4}\sin u = \frac{1}{4}\sin(x^4+2) + C$

#

ex.3

$$\int^{1} \sqrt{2x+1}\;dx$$

$= v = 2x+1 \qquad dv = 2\,dx$

$dx = \dfrac{dv}{2}$

$$= \int \sqrt{v} \cdot \frac{dv}{2}$$

$$= \frac{1}{2} \int \sqrt{v}\;dv$$

$$\frac{1}{\cancel{2}} \cdot \frac{\cancel{2} v^{\frac{3}{2}}}{3} = \frac{v^{\frac{3}{2}}}{3} + c$$

$$= \frac{(2x+1)^{\frac{3}{2}}}{3} + c$$

$\#$

ex4.

$$\int \frac{x}{\sqrt{1-4x^2}}\;dx$$

$v = 1-4x^2 \qquad dv = -8x\;dx$

$dx = \dfrac{dv}{-8x}$

$-\frac{1}{2}+1 \quad \frac{1}{2}$

$\dfrac{v^{\frac{1}{2}}}{\frac{1}{2}}$

$$= \int \frac{\cancel{x}}{\sqrt{v}} \cdot \frac{dv}{-8\cancel{x}}$$

$$= -\frac{1}{8} \int \frac{1}{\sqrt{v}}\;dv = -\frac{1}{8} \int v^{-\frac{1}{2}}$$

$$= -\frac{1}{\cancel{8}} \cdot \cancel{2}\sqrt{v}$$

$$= -\frac{1}{4}\sqrt{1-4x^2} + C\,\#$$

$$\int \cos 5x \; dx$$

$= u = 5x$ $\qquad du = 5 \; dx$

$$dx = \frac{du}{5}$$

$$\int \cos u \; \frac{du}{5}$$

$$= \frac{1}{5} \cdot \sin u + C$$

$$= \frac{1}{5} \cdot \sin 5x + C$$

$$* \int \sqrt{1+x^2} \; x^5 \; dx$$

$u \overset{?}{=} \sqrt{1+x^2}^{\,2}$ $\qquad du = \frac{1}{\cancel{2}}(1+x^2)^{-\frac{1}{2}} \cdot \cancel{2}x \; dx$

$u^2 = 1+x^2$ $\qquad du = \frac{x}{\sqrt{1+x^2}} \; dx$

$(u^2-1)^2 = x^{2^2}$ $\qquad \frac{du \cdot \sqrt{1+x^2}}{x} = \frac{x \; dx}{x}$

$(u^2-1)^2 = x^{\cancel{4}}$

$$dx = \frac{du \cdot \sqrt{1+x^2}}{x}$$

$$= \int u \; \cancel{x^5}^{\,4} \cdot \frac{du \; u}{\cancel{x}}$$

$$= \int u \; x^4 \, du \cdot u$$

$$= \int u^2 x^4 \, du$$

$$= \int u^2 \cdot (u^2-1)^2 \; du$$

$$= \int u^2 \cdot \left[(u^4 - 2u^2 + 1) \right] du$$

$$= \int u^6 - 2u^4 + u^2 \quad du$$

$$= \frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3}$$

$$= \frac{\sqrt{1+x^2}^{\,7}}{7} - \frac{2(\sqrt{1+x^2})^5}{5} + \frac{\sqrt{1+x^2}^{\,3}}{3}$$

$$+ C \cancel{x}$$

Hard way

$$\int_1^2 \frac{dx}{(3-5x)^2}$$

$$u = 3-5x \qquad du = -5\,dx$$

$$dx = \frac{du}{-5}$$

$$= \int_1^2 \frac{1}{u^2} \cdot \frac{du}{-5}$$

$$= -\frac{1}{5} \int_1^2 \frac{1}{u^2}\, du$$

$$-\frac{1}{5} \left. \frac{u^{-1}}{-1} \right|_1^2$$

$$= -\frac{1}{5} \left[\left. -\frac{1}{3-5x} \right|_1^2 \right]$$

$$= -\frac{1}{5} \left[\left(-\frac{1}{3-10} \right) - \left[-\frac{1}{3-5} \right] \right] \qquad \frac{2-7}{14} = -\frac{\cancel{8}}{14} \cdot -\frac{1}{\cancel{5}}$$

$$= -\frac{1}{5} \left[\frac{1}{7} - \frac{1}{2} \right] = \frac{1}{14}$$

$$\cancel{7}$$

Integral of symmetric function

Integration by parts

$\int u\,dv = uv - \int v \cdot du$

ex.1 $x \sin x \quad dx$

$u = x \qquad du = dx$

$dv = \sin x \ dx \qquad v = -\cos x$

$= -x\cos x - \int -\cos x \ dx$

$-x\cos x + \sin x + C$ #

$\int \ln(x)\, dx$

$u = \ln(x) \qquad du = \frac{1}{x}\, dx$

$dv = dx \qquad v = x$

$x\ln(x) - \int \frac{1}{\cancel{x}} \cdot \cancel{x}\, dx$

$x\ln(x) - x + C$ #

$\int t^2 e^t \, dt$

$u = t^2 \qquad du = 2t \ dt$

$dv = e^t \, dt \qquad v = e^t$

$= t^2 \cdot e^t - \int e^t \cdot 2t \ dt$

$\downarrow$

$u = 2t \quad du = 2\ dt$

$dv = e^t \ dt \qquad v = et$

$= -\left[2t \cdot e^t - 2\int e^t \ dt\right)$

$= -\left[2te^t - 2e^t\right]$

$\text{Ans} = t^2e^t - 2te^t + 2e^t + C$ #

$$\int \frac{1}{\cos^3(x)}\,dx$$

$$= \int \frac{1}{\cos^2 x}\cdot\frac{1}{\cos(x)}\,dx$$

$\downarrow$

$$\int \sec^2 x \cdot \frac{1}{\cos(x)}\,dx$$

$$u = \frac{1}{\cos(x)} \qquad du = -\cos(x)^{-2} \times -\sin x$$

$$= \frac{\sin x}{(\cos(x))^2}$$

$$dv = \sec^2 x\,dx \qquad v = \tan x \longrightarrow \frac{\sin x}{\cos(x)}$$

$$= \frac{1}{\cos(x)}\cdot\frac{\sin x}{\cos(x)} - \int \frac{\sin x}{\cos(x)}\cdot\frac{\sin x}{\cos^2(x)}$$

$$= \frac{\sin x}{\cos^2(x)} - \int \frac{\sin^2 x}{\cos^3(x)} \qquad \sin^2 x = 1 - \cos^2 x$$

$\downarrow$

$$\int \frac{1-\cos^2 x}{\cos^3 x}$$

let $\int \frac{1}{\cos^3(x)} = I$

$$= \int \frac{1}{\cos^3(x)}\,dx - \int \frac{1}{\cos(x)}\,dx$$

$$I = \frac{\sin x}{\cos^2(x)} - I +$$

$\downarrow$ Integrate $\int \frac{1}{\cos(x)}\,dx$

$\downarrow$

$$= \int \sec(x)\, dx$$

$$\downarrow$$

multiply conjugate

$$= \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx$$

$$\downarrow$$

$$v = \sec x + \tan x \qquad dv = \sec x \tan x + \sec^2 x$$

$$dx = \frac{dv}{\sec x \tan x + \sec^2 x}$$

$$= \int \frac{\cancel{\sec^2 x + \sec x \tan x}}{v} \cdot \frac{dv}{\cancel{\sec x \tan x + \sec^2 x}}$$

$$= \int \frac{1}{v}\, dv = \ln(\sec x + \tan x).$$

$$\text{final} = I = \frac{\sin x}{\cos^2(x)} - I + \ln(\sec x + \tan x)$$

$$\frac{2I = \dfrac{\sin x}{\cos^2(x)} + \ln(\sec x + \tan x)}{2}$$

$$I = \frac{1}{2}\left(\frac{\sin x}{\cos^2(x)} + \ln(\sec x + \tan x) \right) + C$$

$$\int \cos(\ln(x))\,dx$$

$u = \cos(\ln(x)) \qquad du = -\sin(\ln(x)) \cdot \frac{1}{x}\,dx$

$dv = dx \qquad v = x$

$$= x\cos(\ln(x)) - \int \cancel{x} \cdot -\sin(\ln(x)) \cdot \frac{1}{\cancel{x}}\,dx$$

$$= x\cos(\ln(x)) + \int \sin(\ln(x))\,dx$$

$\downarrow$

$u = \sin(\ln(x)) \qquad du = \cos(\ln(x)) \cdot \frac{1}{x}\,dx$

$dv = dx \qquad v = x$

$x\sin(\ln(x)) - \int \cancel{x} \cdot \frac{1}{\cancel{x}} \cos(\ln(x))\,dx$

$x\sin(\ln(x)) - \int \cos(\ln(x))\,dx$

let $I = \int \cos(\ln(x))\,dx$

$$I = x\cos(\ln(x)) + x\sin(\ln(x)) - \underline{I}$$

$$2I = x\cos(\ln(x)) + x\sin(\ln(x))$$

$$I = \frac{1}{2}\left(x\cos(\ln(x)) + x\sin(\ln(x))\right) + C$$

$\#$

$$\int \sqrt{x} \ln(x) \, dx$$

$$u = \ln(x) \qquad du = \frac{1}{x} \, dx$$

$$dv = \sqrt{x} \, dx \qquad v = \frac{2x^{\frac{3}{2}}}{3}$$

$$= \frac{2x^{\frac{3}{2}}}{3} \cdot \ln(x) - \int \frac{2x^{\frac{3}{2}}}{3} \cdot \frac{1}{x} \, dx$$

$$\downarrow$$

$$\int \frac{2x^{\frac{1}{2}}}{3} \, dx$$

$$- \frac{2}{3} \int x^{\frac{1}{2}} \, dx$$

$$- \frac{2}{3} \cdot \left(\frac{2x^{\frac{3}{2}}}{3} \, dx \right)$$

$$= \frac{2}{3} x^{\frac{3}{2}} \ln(x) - \frac{4x^{\frac{3}{2}}}{9} + C_{\cancel{x}}$$

$$\int x \ln(x)^2 \, dx$$

$$u = \ln(x)^2 \qquad du = 2 \ln(x) \cdot \frac{1}{x} \, dx$$

$$dv = x \, dx \qquad v = \frac{x^2}{2}$$

$$= \frac{x^2}{2} \cdot \ln(x)^2 - \int \frac{x^{\cancel{2}}}{\cancel{2}} \cdot \frac{\cancel{2}\ln(x)}{\cancel{x}} \, dx$$

$$= \frac{x^2}{2}\cdot \ln(x)^2 - \int x \ln(x)\, dx$$

$\downarrow$

$u = \ln(x) \qquad du = \frac{1}{x}\, dx$

$dv = x\, dx \qquad v = \frac{x^2}{2}$

$$= -\left[\frac{x^2}{2}\ln(x) - \int \frac{x^{\cancel{2}}}{2}\cdot\frac{1}{\cancel{x}}\, dx \right.$$

$$-\left[\frac{x^2}{2}\ln(x) - \frac{1}{2}\int x \, dx \right.$$

$$-\left[\frac{x^2}{2}\ln(x) - \frac{x^2}{4} \right] + c$$

$$= \frac{x^2}{2}\cdot \ln(x)^2 - \frac{x^2}{2}\ln(x) + \frac{x^2}{4} + C$$

Substitution

$$= \int \frac{1}{1+\sqrt{1+x}}\, dx$$

$u = \sqrt{1+x} \qquad du = \frac{1}{2}(1+x)^{-\frac{1}{2}}\cdot dx$

$dx = 2\sqrt{1+x}\, du$

$$= \int \frac{1}{1+u}\cdot 2u\, du$$

$$= \int \frac{2u}{1+u}\, du = 2\int \frac{u}{1+u} = 2\int \frac{u+1-1}{u+1} =$$

$\downarrow$

$$= 2\int 1 - \int \frac{1}{u+1}$$

$$= 2\left[x - \ln|u+1| \right]$$

$$= 2x - 2\ln|u+1| + c$$

$$= 2\sqrt{1+x} - 2\ln\left|\sqrt{1+x} + 1\right| + c$$

✓

$$\int e^x \sin x \, dx$$

$=$ $u = e^x$ $\qquad du = e^x \, dx$

$dv = \sin x \, dx \qquad v = -\cos x$

$$= -e^x \cos x - \int -\cos x \, e^x \, dx$$

$$= -e^x \cos x + \int \cos x \, e^x \, dx$$

$u = e^x \qquad du = e^x \, dx$

$dv = \cos x \qquad v = \sin x$

$$= e^x \sin x - \int \sin x \, e^x \, dx$$

$$I = -e^x \cos x + e^x \sin x - I$$

$$2I = -e^x \cos x + e^x \sin x$$

$$I = \frac{1}{2}\left(-e^x \cos x + e^x \sin x\right)$$

$$\int_0^1 \tan^{-1} x \, dx$$

$$= \quad v = \tan^{-1} y \qquad dv = \frac{1}{1+x^2}\, dx$$

$$dv = dx \qquad v = x$$

$$= \frac{1}{\tan x} \cdot x - \int \frac{x}{1+x^2}\, dx$$

$$\downarrow$$

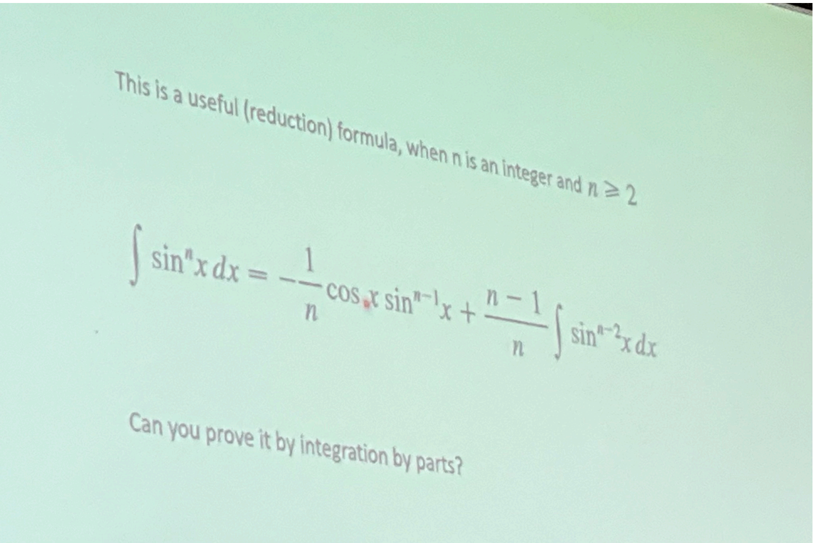
$$v = 1+x^2 \qquad dv = 2x\, dx$$

$$dx = \frac{dv}{2x}$$

$$= \int \frac{\cancel{x}}{v} \cdot \frac{dv}{2\cancel{x}}$$

$$= \frac{1}{2} \ln |1+x^2)$$

$$= \frac{x}{\tan x} - \frac{1}{2} \ln(1+x^2) \Big|_0^1$$



This is a useful (reduction) formula, when n is an integer and $n \geq 2$

$$\int \sin^n x \, dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

Can you prove it by integration by parts?

$$\int \cos^3 x \quad dx$$

$$= \int \cos^2 x \cdot \cos x \ dx$$

↓

$$= \int (1-\sin^2 x)\cdot \cos x \ dx$$

$u = 1-\sin^2 x \qquad du = -2(\sin x)\cdot \cos x \ dx$

$dv = \cos x \ dx \qquad v = \sin x$

$$= \sin x(1-\sin^2 x) - \int \sin x \cdot -2\sin x \cos x$$

$$= \sin x(1-\sin^2 x) - \int -2\sin^2 x \cos x$$

↓

$$\int -2(1-\cos^2 x)\cos x$$

$$= -2\int \cos x - \int \cos^3 x$$

$$= \sin x(1-\sin^2 x) + 2[\sin x - I]$$

$$I = \sin x(1-\sin^2 x) + 2\sin x - 2I$$

$$\frac{3I = \sin x(1-\sin^2 x) + 2\sin x}{3}$$

$$= \frac{1}{3}\left(\sin x(1-\sin^2 x) + 2\sin x\right)$$

$$= \frac{1}{3}(\sin x - \sin^3 x + 2\sin x) = \frac{1}{3}(3\sin x - \sin^3 x) + C$$

#

Second ways

let $u = \sin x$

$du = \cos x \ dx$

$dx = \frac{du}{\cos x}$

$$= \int (1-u^2)\cdot \cancel{\cos x}\left(\frac{du}{\cancel{\cos x}}\right)$$

$$= \int 1-u^2 \ du$$

$$= u - \frac{u^3}{3} \ d$$

$$= \sin x - \frac{1}{3}\sin^3 x + C$$

$\# \int \sin^5 x \cos^2 x \, dx$

$= \int (\sin^2 x)^2 \cdot \cos^2 x \cdot \sin x \, dx$

$= \int (1-\cos^2 x)^2 \cdot \cos^2 x \cdot \sin x \, dx$

let $v = \cos x$ $\quad dv = -\sin x \, dx$

$dx = \frac{dv}{-\sin x}$

$= \int (1-v^2)^2 \cdot v^2 \cdot \cancel{\sin x} \cdot \frac{dv}{-\cancel{\sin x}}$

$-1 \int (v^4 - 2v^2 + 1) \cdot v^2$

$= -1 \int v^6 - 2v^4 + v^2 = -\left(\frac{v^3}{3} - 2\frac{v^5}{5} + \frac{v^7}{7}\right) + C$

$= -\left(\frac{\cos x^3}{3} - \frac{2(\cos x)^5}{5} + \frac{\cos x^7}{7}\right) + C \quad \#$

Rule # 1

Odd powers of sine or cosine allows us to:

1) separate a single factor.

2) Convert the remaining even power.

Ex. ถ้าเจอ $\int \sin^7 x \cos^4 x \, dx$

$\downarrow$

ดึง $\sin x$ ออกมาเป็น factor ตัวเดียว

$= \int (\sin^2 x)^3 \cos^4 x \cdot \sin x \, dx$ #

!! However

If the integrand contains even powers of both sine and cosine, this strategy then fails.

Solution: use half-angle identities $\longrightarrow$ $\sin^2 x = \frac{1}{2}(1-\cos 2x)$, $\cos^2 x = \frac{1}{2}(1+\cos 2x)$

$$\int_0^{\pi} \sin^2 x \ dx \Bigg\}$$ even power → so use half-angle identities.

$$= \int_0^{\pi} \frac{1}{2}(1 - \cos 2x) \ dx$$

$$= \frac{1}{2}\int_0^{\pi} 1 - \cos 2x \ dx$$

$$\downarrow$$

$$v = 2x \qquad dv = 2\,dx$$

$$dx = \frac{dv}{2}$$

$$\frac{1}{2}\int_0^{\pi} 1 - \cos v \cdot \frac{dv}{2}$$

$$= \frac{1}{4}\int_0^{\pi} 1 - \cos v$$

$$= \frac{1}{4}\Big[v - \sin v \Big]_0^{\pi}$$

$$= \frac{1}{4}\Big[2x - \sin 2x \Big|_0^{\pi} \Big]$$

$$= \frac{1}{4}\Big[[2(\pi) - \sin 2\pi] - [2(0) - \sin(0)] \Big]$$

$$= \frac{1}{4}\Big[(6.17) - (0) \Big] = \frac{1}{4}(6.17) = \frac{1}{2}\pi$$

$\cancel{\cancel{t}}$

$$\int \sin^4 x \; dx$$

$$= \int (\sin^2 x)^2 \; dx \qquad = \sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$= \int \left(\frac{1}{2}(1 - \cos 2x)\right)^2 \; dx$$

$$= \int \left(\frac{1 - \cos 2x}{2}\right)^2 \; dx$$

$$= \int \left(\frac{1}{2} - \frac{\cos 2x}{2}\right)^2 \longrightarrow \left(\frac{1}{2} - \frac{\cos 2x}{2}\right)\left(\frac{1}{2} - \frac{\cos 2x}{2}\right)$$

$$= \frac{1}{4}\int 1 - 2\cos 2x + \cos^2 2x \; dx \qquad \frac{1}{4} - \frac{\cos 2x}{4} - \frac{\cos 2x}{4} + \frac{\cos^2 2x}{4}$$

$$= \frac{1}{4} - \frac{2\cos 2x}{4} + \frac{\cos^2 2x}{4}$$

$v = 2x \qquad dv = 2\,dx$

$dx = \frac{dv}{2}$

$$= \frac{1}{4}\int 1 - 2\cos v + \cos^2 v \cdot \frac{dv}{2}$$

$$= \frac{1}{8}\int 1 - 2\cos v + \cos^2 v \;\; dv \qquad = \int \cos^2 x \;\; dx$$

$$= \frac{1}{8}\left[v - 2\sin v + \frac{1}{2}v + \frac{1}{4}\sin 2v\right] \qquad \int \frac{1}{2}(1 + \cos 2x)$$

$$= \frac{1}{8}\left[2x - 2\sin 2x + x + \frac{1}{4}\sin 4x\right] = \qquad \int \frac{1}{2} + \frac{\cos 2x}{2} \; dx$$

$$= \frac{1}{2}x + \frac{1}{4}(\sin 2x)$$

$$= \frac{1}{8}\left[3x + \frac{1}{4}\sin 4x - 2\sin 2x\right]$$

$\int \cos 2x \;\; dx$

$v = 2x \qquad dv = 2\,dx$

$dx = \frac{dv}{2}$

$= \int \cos v \cdot \frac{dv}{2}$

$= \frac{1}{2}\sin 2x$

Rule # 2 → Continued from rule # 1.

$\int \sin^m x \cos^n x \, dx$, where $m \geq 0$ and $n \geq 0$ are integers.

a) If the power of cosine is odd ($n = 2k+1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaing factors interm of sine.

ex. $\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x \, (\cos^2 x)^k \cos x \, dx$

$$= \int \sin^m x \, (1 - \sin^2 x)^k \cos x \, dx$$

Then sub → $u = \sin x$.

b) If the power of sine is odd ($m = 2k+1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in term of cosine:

$$\int \sin^{2k+1} x \cos^n x \, dx = \int (\sin^2 x)^k \cos^n x \sin x \, dx$$

$$= \int (1 - \cos^2 x)^k \cos^n x \sin x \, dx$$

then sub → $u = \cos x$

3) If both even then use double angle - Identities.

Rule # 3

$$\int \tan^m x \sec^n x \; dx$$

$\left(\frac{d}{dx}\right) \tan x = \sec^2 x$, so

!! If the power of secant is even.

1) $\sec^2 x$ can be factor

2) the remaining (even) power of sec can be converted to an expression involving tangent using $\sec^2 x = 1 + \tan^2 x$

Or $\left(\frac{d}{dx}\right) \sec x = \sec x \tan x$, so

!! If the power of tangent is odd.

1) a sec x tan x factor can be separated and.

2) the remaining (even) power of tangent can be convert to secant.

Ex.1

$$\int \tan^6 x \sec^4 x \; dx$$

$$\int (\tan^6 x)(\sec^2 x)(\sec^2 x)\; dx$$

$$= \int (\tan^6 x)(1 + \tan^2 x)(\sec^2 x)\; dx$$

$v = \tan x$ $\quad dv = \sec^2 x \; dx$

$dx = \frac{dv}{\sec^2 x}$

$$= \int (v)^6 (1 + v^2) \cdot \cancel{\sec^2 x} \cdot \frac{dv}{\cancel{\sec^2 x}}$$

$$= \int v^6 + v^8 \; dv = \frac{v^7}{7} + \frac{v^9}{9} + C = \frac{(\tan x)^7}{7} + \frac{(\tan x)^9}{9} + C$$ #

Ex, 2

$$\int \tan^5\theta \sec^7\theta \, d\theta$$

$$= \int \tan^4\theta \sec^6\theta \, (\sec\theta \tan\theta)$$

$$= \int (\tan^2\theta)^2 \sec^6\theta \, (\sec\theta \tan\theta) \, d\theta$$

$$= \int (\sec^2\theta - 1)^2 (\sec^6\theta)(\sec\theta \tan\theta) \, d\theta$$

$u = \sec\theta$ $\quad du = \sec\theta \tan\theta \, d\theta$

$$d\theta = \frac{du}{\sec\theta \tan\theta}$$

$$= \int (u^2-1)^2 (u)^6 \cdot \cancel{\sec\theta \tan\theta} = \frac{du}{\cancel{\sec\theta \tan\theta}}$$

$$= \int (u^2-1)^2 (u)^6 \, du$$

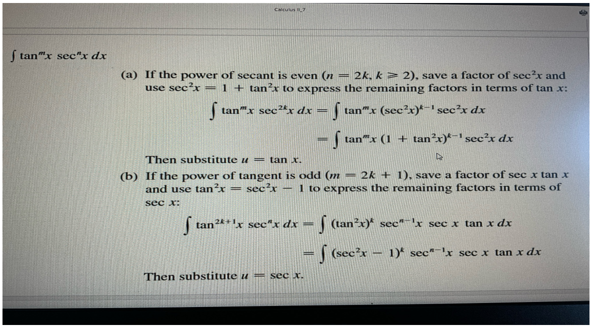
$$= \int (u^4 - 2u^2 + 1)(u)^6 \, du$$

$$= \int u^{10} - 2u^8 + u^6 \, du = \frac{u^{11}}{11} - \frac{2u^9}{9} + \frac{u^7}{7}$$

$$= \frac{(\sec\theta)^{11}}{11} - \frac{2(\sec\theta)^9}{9} + \frac{(\sec\theta)^7}{7} + C$$

#

Full definition



Calculus II_7

$\int \tan^m x \sec^n x \, dx$

(a) If the power of secant is even ($n = 2k$, $k \geqslant 2$), save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:

$$\int \tan^m x \sec^{2k} x \, dx = \int \tan^m x \, (\sec^2 x)^{k-1} \sec^2 x \, dx$$

$$= \int \tan^m x \, (1 + \tan^2 x)^{k-1} \sec^2 x \, dx$$

Then substitute $u = \tan x$.

(b) If the power of tangent is odd ($m = 2k + 1$), save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$$\int \tan^{2k+1} x \sec^n x \, dx = \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x \, dx$$

$$= \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x \, dx$$

Then substitute $u = \sec x$.

Two useful formulas:

$$\int \tan x \, dx = \ln|\sec x| + C$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

EX. $\int \tan^3 x \, dx.$

$= \int \tan^2 x \cdot \tan x \, dx$

$= \int (\sec^2 x - 1) \tan x \, dx$

$= \int \tan x \sec^2 x - \tan x \, dx$

$= \int \tan x \sec^2 x \, dx - \int \tan x \, dx$

$\downarrow$

$u = \tan x$

$du = \sec^2 x \, dx$

$dx = \dfrac{du}{\sec^2 x}$

$= \int u \cdot \cancel{\sec^2 x} \cdot \dfrac{du}{\cancel{\sec^2 x}}$

$= \int u \, du$

$= \dfrac{u^2}{2}$

$\int \operatorname{Tan} x = \ln|\sec x| + c$

$= \frac{1}{2} \tan x^2 - \ln|\sec x| + C$

Ex. 2

$$\int \sec^3 x \; dx$$

$$= \int \sec^2 x \cdot \sec x \quad dx$$

$u = \sec x \qquad du = \sec x \tan x$

$dv = \sec^2 x \qquad v = \tan y$

$$= \sec x \tan x - \int \tan x \cdot \sec x \tan x \; dx$$

$$\downarrow$$

$$- \int \sec x \tan^2 x \quad dx$$

$$- \int \sec x \left(\sec^2 x - 1 \right) dx$$

$$- \int \sec^3 x - \sec x \quad dx$$

$$I = \sec x \tan x - \left[I - \int \sec x \right]$$

$$I = \sec x \tan x - I + \ln|\sec x + \tan x|$$

$$2I = \sec x \tan x + \ln|\sec x + \tan x|$$

$$I = \frac{1}{2} \left(\sec x \tan x + \ln|\sec x + \tan x| \right)$$

Rule # 4

for (a) $\int \sin mx \cos nx \, dx$, (b) $\int \sin mx \sin nx \, dx$, (c) $\int \cos mx \cos nx \, dx$,

we can use.

(a.) $\sin A \cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$

(b) $\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

(c) $\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$

Evaluate $\int \sin 4x \cos 5x \, dx$

$$= \int \frac{1}{2}\left[\sin(4x-5x) + \sin(4x+5x)\right]$$

$$= \int \frac{1}{2}\left[\sin(-x) + \sin(9x)\right]$$

$$= \frac{1}{2}\left[\int - \sin x \, nx + \int \sin(9x) \, dx \right]$$

$$= \frac{1}{2}\left[+\cos x - \frac{1}{9}\cos(9x)\right] + C$$

#

Trig Sub

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta,\ 0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

$$\int \frac{\sqrt{9-x^2}}{x^2}\, dx$$

$x = 3\sin\theta \qquad dx = 3\cos\theta\ d\theta$

$$= \sqrt{9 - 9\sin^2\theta} = \sqrt{9(1-\sin^2\theta)} = \sqrt{9\cos^2\theta} = 3\cos\theta$$

$$= \int \frac{3\cos\theta}{9\sin^2\theta} \cdot 3\cos\theta\ d\theta$$

$\frac{\cos}{\sin} = \cot$

$$= \int \frac{9\cos^2\theta}{9\sin^2\theta}\, d\theta = \int \frac{\cos^2\theta}{\sin^2\theta}\, d\theta = \int \cot^2\theta\ d\theta$$

Trig Identity

จำเอา

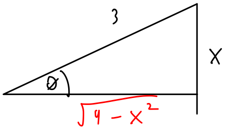
$$\cot^2\theta = \csc^2\theta - 1$$

$$= \int (\csc^2\theta - 1)\, d\theta \longrightarrow \text{Integrate of } \csc^2\theta$$

$$= -\cot\theta - \theta + C \longrightarrow \text{Remember}$$

↓

as $\sin\theta = \frac{x}{3}$



$$a^2 + x^2 = 3^2$$

$$a^2 = 3^2 - x^2$$

$$\sqrt{a^2} = \sqrt{9 - x^2}$$

$$a = \sqrt{9 - x^2}$$

$$\boxed{\cot = \frac{1}{\tan x}}$$

$$\tan\theta = \frac{x}{\sqrt{9 - x^2}}$$

Thus $\cot\theta = \frac{\sqrt{9 - x^2}}{x}$

$$\text{Ans} = -\frac{\sqrt{9 - x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$$

#

Ex.2 $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$

$x = 2\tan\theta$ $dx = 2\sec^2\theta\,d\theta$

$$\sqrt{(2\tan\theta)^2 + 4} = \sqrt{4\tan^2\theta + 4} = \sqrt{4(\tan^2\theta + 1)} = \sqrt{4\sec^2\theta}$$

$$= 2\sec\theta$$

$$= \int \frac{1}{4\tan^2\theta \cdot \cancel{2}\sec\theta} \cdot \cancel{2}\sec^2\theta\,d\theta = \frac{1}{4}\int \frac{\sec^{\cancel{2}}\theta\,d\theta}{\tan^2\theta \cdot \cancel{\sec\theta}} = \frac{1}{4}\int \frac{\sec\theta\,d\theta}{\tan^2\theta}$$

↓

$$\frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta$$

$$= \frac{1}{4}\int \frac{1}{\cancel{\cos\theta}} \cdot \frac{\cos^{\cancel{2}}\theta}{\sin^2\theta}$$

$$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta$$

$$v = \sin\theta \qquad dv = \cos\theta\ d\theta$$

$$= \frac{1}{4}\int \frac{\cancel{\cos\theta}}{v^2} \cdot \frac{dv}{\cancel{\cos\theta}}$$

$$= \frac{1}{4}\int \frac{1}{v^2}\, dv$$

$$= \frac{1}{4}\left[-1\ v^{-1} \right]$$

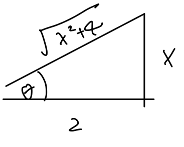
$$= \frac{-1}{4} \cdot \frac{1}{\sin\theta}$$

$$= -\frac{1}{4\sin\theta} + C$$

as $x = 2\tan\theta$

$$\tan\theta = \frac{x}{2}$$

=



$$\sin\theta = \frac{x}{\sqrt{x^2+4}}$$

$$= \frac{-1}{4\left(\frac{x}{\sqrt{x^2+4}}\right)} = \frac{-1}{\frac{4x}{\sqrt{x^2+4}}} = -\frac{\sqrt{x^2+4}}{4x} + C$$

$$\int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{(4x^2+9)^{\frac{3}{2}}}\,dx$$

$$\frac{x^3}{(4x^2+9)^{\frac{3}{2}}}$$

$$\frac{x^3}{\left(\sqrt{\frac{4x^2+9}{4}}\right)(4x^2+9)}$$

$$\frac{x^3}{\left(\sqrt{4x^2+9}\right)(4x^2+9)}$$

$$= \frac{x^3}{\left(\sqrt{x^2+\frac{9}{4}}\right)(4x^2+9)}\,dx$$

$x = \frac{3}{2}\tan\theta$ $\quad dx = \frac{3}{2}\sec^2\theta\,d\theta$

$$\downarrow$$

$$\left(\frac{3}{2}\right)^2$$

use $a\tan\theta$

sub back to old equation

let $x = \frac{3}{2}\tan\theta$ $\quad dx = \frac{3}{2}\sec^2\theta \quad d\theta$

$$= \sqrt{\left(4\left(\frac{9}{4}\tan^2\theta\right)+9\right.} = \sqrt{9\tan^2\theta+9} = \sqrt{9(\tan^2\theta+1)} = \sqrt{9\sec^2\theta} = 3\sec\theta$$

$$4x^2+9 = 4\left(\frac{3}{2}\tan\theta\right)^2+9 = 4\left(\frac{9}{4}\tan^2\theta\right)+9 = 9\tan^2\theta+9 = 9(\tan^2\theta+1)$$

$$= 9\sec^2\theta$$

$$= \int \frac{\frac{\cancel{27}^{\,3}}{8}\tan^3\theta}{(\cancel{3}\sec\theta)(\cancel{9}\sec^2\theta)} \cdot \frac{\cancel{3}}{2}\sec^2\theta \quad d\theta$$

$$\frac{3}{16}\int \frac{(\sin^2\theta)(\sin\theta)}{\cos^2\theta}\,d\theta =$$

$$= \frac{3}{16}\int \frac{\tan^3\theta}{\sec\theta}\,d\theta$$

$V = \cos\theta$

$$\boxed{= \frac{3}{16}\int \frac{(1-\cos^2\theta)(\sin\theta)}{\cos^2\theta}\,d\theta}$$

$$= \frac{3}{16}\int \frac{\frac{\sin^3\theta}{\cos^3\theta}}{\frac{1}{\cos\theta}}\,d\theta$$

$$\downarrow$$

$v = \cos\theta \qquad dv = -\sin\theta \quad d\theta$

$$= \frac{3}{16}\int \frac{\sin^3\theta}{\cos^2\theta}\,d\theta$$

$$= \frac{3}{16}\int \frac{(1-v^2)}{v^2} \cdot \sin\cancel{\theta} \cdot \frac{dv}{-\cancel{\sin\theta}}$$

$$\downarrow$$

$$= -\frac{3}{16}\int \frac{1-v^2}{v^2}$$

$$= -\frac{3}{16}\left[\int \frac{1}{v^2} - \int 1 \right]$$

$$= -\frac{3}{16}\left[-\frac{1}{v} - v \right]$$

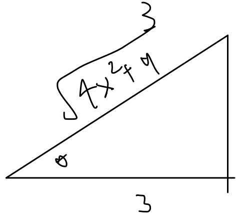
$$\cancel{\times}$$

$$\boxed{= +\frac{3}{16}\left[\frac{1}{\cos\theta} + \cos\theta \right]} \Bigg|_{0}^{\frac{\pi}{3}}$$

$$\tan\theta = \frac{2x}{3}$$

$$\frac{\sqrt{\ }}{2} - 2$$

$$= \sqrt[3]{2}$$



$$\cos\theta = \frac{3}{\sqrt{4x^2+9}}$$

$$= \frac{3}{16}\left[\frac{1}{\frac{3}{\sqrt{4x^2+9}}} - \frac{3}{\sqrt{4x^2+9}} \right]$$

$$= \frac{3}{16}\left[\frac{\sqrt{4x^2+9}}{3} - \frac{3}{\sqrt{4x^2+9}} \right]$$

$$= \frac{\sqrt{4x^2+9}}{16} + \frac{9}{16\sqrt{4x^2+9}} \Bigg|_{0}^{\frac{3\sqrt{3}}{2}}$$

$$\int$$

Calculate definite integral to get $\frac{32}{3}$ #

When to use trigonometric substitution !!

Use when we have $(x^2+a^2)^{\frac{n}{2}}$ in an integral, where n is any integer.

The same is true when we have $(a^2-x^2)^{\frac{n}{2}}$ or $(x^2-a^2)^{\frac{n}{2}}$

$$= \int \frac{x}{\sqrt{3-2x-x^2}}\,dx$$

} Quite tricky. ✱

complete the square

$$= -x^2-2x+3 = x^2+2x-3$$

$$x^2+2x+\left(\frac{6}{2}\right)^2 = 3+$$

$$x^2+2x+1 = 3+1$$

$$x^2+2x+1 = 4$$

$$-\left[(x+1)^2-4=\right]$$

$$= -(x+1)^2+4$$

$$= \int \frac{x}{\sqrt{4-(x+1)^2}}\,dx$$

let $v = x+1$

$$\therefore v = 2\sin\theta \qquad dv = 2\cos\theta\,d\theta$$

$$\sqrt{4-v^2} = \sqrt{4-(2\sin\theta)^2}$$

$$= \sqrt{4-4\sin^2\theta}$$

$$= \sqrt{4(1-\sin^2\theta)} = \sqrt{4\cos^2\theta}$$

$$= 2\cos\theta$$

$$= \int \frac{2\sin\theta - 1}{\cancel{2\cos\theta}} \cancel{2\cos\theta} \, d\theta$$

$$= \int 2\sin\theta - 1$$

$$= \int 2\sin\theta - 1$$

$$= -2\cos\theta - \theta + C$$

$$= -\sqrt{4 - v^2} - \sin^{-1}\left(\frac{v}{2}\right)$$

$$\sqrt{3 - 2x - x^2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C$$

$$= -$$

Integration of Rational Functions

$f(x) = \dfrac{P(x)}{Q(x)}$ → In a proper rational function, the deg (P) is smaller than deg (Q).

If f is improper (deg (P) > deg (Q)), then we must first divide Q into P

Ex. → $f(x) = \dfrac{P(x)}{Q(x)} = S(x) + \dfrac{R(x)}{Q(x)}$

Ex1. $\displaystyle\int \frac{x^3 + x}{x-1}\,dx.$

$$\begin{array}{r|l} & x^2 + x + 2 \\ \hline x-1 & x^3 \quad 0 \quad x \\ & x^3 - x^2 \\ & \quad x^2 + x \\ & \quad x^2 - x \\ & \qquad 2x \\ & \qquad 2x - 2 \\ & \qquad\quad 2 \end{array}$$

$$= \int x^2 + x + 2 + \frac{2}{x-1}$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ln|x-1) + C$$

But we may not always face with simple denominators. !!!

Case 1 : Where no factor is repeated or is a constant multiple of another.

$$Q(x) = (a_1 x + b_1)(a_2 x + b_2) \cdots (a_k x + b_k)$$

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1 x + b_1} + \frac{A_2}{a_x 2 + b_2} + \cdots + \frac{A_k}{a_k x + b_k}$$

$$\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} \, dx$$

!

$$2x^3 + 3x^2 - 2x = x(2x^2 + 3x - 2) = x(2x-1)(x+2)$$

$$= \left(\frac{x^2 + 2x - 1}{\cancel{x(2x-1)(x+2)}} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2} \right) x(2x-1)(x+2)$$

$$x^2 + 2x - 1 = A(2x-1)(x+2) + B(x)(x+2) + C(x)(2x-1))$$

$$= \cancel{A}(\cancel{2x^2} + \cancel{3x} - 2) + \cancel{Bx^2} + \cancel{2Bx} + \cancel{2Cx^2} - Cx$$

$$x^2(2A + B + 2C) + x(3A + 2B - C) - 2A$$

$$-2A = -1$$

$$A = \frac{1}{2}$$

$3A + 2B - \cancel{C} = 2$ $\qquad\qquad$ $L - \frac{3}{2}$

$2A \quad + B + 2C = 1$

$2\left(2B - C = \frac{1}{2}\right)$

$B + 2C = 0$

$4B - 2C = 1$ $\qquad\qquad$ $C = -\frac{1}{10}$

$B + 2C = 1$

$5B = 1$

$B = \frac{1}{5}$

$$= \int \frac{1}{2}\frac{1}{x} + \frac{1}{5}\frac{1}{2x-1} - \frac{1}{10}\frac{1}{x+2} \Big) \, dx$$

$$= \frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + K \quad \#$$

Ex 2. find $\int \frac{1}{x^2-a^2}\,dx$

$$= \left[\frac{1}{x^2 - a2} = \frac{1}{\cancel{(x+a)}\cancel{(x-a)}} = \frac{A}{(x+a)} + \frac{B}{(x-a)} \right] (x+a)(x-a)$$

$$1 = A(x-a) + B(x+a)$$

$x = a \qquad 1 = 2aB$

$B = \frac{1}{2a}$

$x = -a \qquad A = -\frac{1}{2a}$

$$= \int -\frac{1}{2a}\left(\frac{1}{x+a}\right) + \frac{1}{2a}\left(\frac{1}{x-a}\right)$$

$$= \frac{1}{2a} \int \left(\frac{1}{(x-a)} - \frac{1}{(x+a)} \right)$$

$$= \frac{1}{2a} \left(\ln|x-a| - \ln|x+a| \right) + C \quad \#$$

$$\downarrow$$

$$= \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C$$

Case 2 Suppose we have the factor $(a_1x + b_1)$ being r times.

Then we use.

$$\frac{A_1}{a_1x+b_1} + \frac{A_2}{(a_1x+b_1)^2} + \cdots + \frac{A_r}{(a_1x+b_1)^r}$$

Ex.

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} \, dx.$$

$$\begin{array}{r} x+1 \\ x^3-x^2-x+1 \,\overline{)\, x^4 + 0 - 2x^2 + 4x + 1} \\ x^4 - x^3 - x^2 + x \\ \cancel{x^3} - x^2 + 3x + 1 \\ \cancel{x^3} - x^2 - x + 1 \\ 4x + 0 \end{array}$$

$$= \int x + 1 + \frac{4x}{x^3 - x^2 - x + 1}$$

$$\downarrow$$

$$\frac{4x}{x^3-x^2-x+1} = \quad \longrightarrow \quad x^2(x-1)-(x-1) = (x^2-1)(x-1)$$

$$= \left[\frac{4x}{(x^2-1)(x-1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} \right]$$

x^2-1

$\downarrow$

$= (x+1)(x-1)$

$= x^2-x+x-1$

$= x^2-1$

$A + C = 0 \qquad C = -A$

$B - 2C = 4$

$-A + B + C = 0$

$B - 2C = 4$

$B + 2C = 0$

$2B = 4$

$B = 2$

$2 - 2C = 4$

$-2C = 2$

$C = -1$

$A = 1$

$$= \int \left[x + 1 + \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1} \right] dx$$

$$= \frac{x^2}{2} + x + \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + K$$

Case 3

If the $Q(x)$ has the factor ax^2+bx+c, where $b^2-4ac<0$, then a term of this form also will appear.

$$\frac{Ax+B}{ax^2+bx+c}$$

Useful Formula : $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+C$

Ex. find $\int \frac{2x^2-x+4}{x^3+4x}\,dx$

$$\downarrow$$

$$= \left[\frac{2x^2-x+4}{x\cancel{(x^2+4)}} = \frac{A}{x} + \frac{Bx+C}{x^2+4}\right] \cancel{(x)}(x^2+4)$$

$$2x^2-x+4 = A(x^2+4) + (Bx+C)(x)$$

$$= Ax^2+4A+Bx^2+Cx$$

$$= x^2(A+B)+Cx+4A$$

$$C=-1,\quad A=1,\quad B=1 \qquad A+B=2$$

$$= \int \frac{1}{x} + \frac{x-1}{x^2+4}\,dx$$

$$= \ln|x| + \int \frac{x-1}{x^2+4}\,dx$$

$$\downarrow$$

$$\int \frac{x}{x^2+4}\,dx - \int \frac{1}{x^2+4}$$

$\downarrow$

$v = x^2 + 4 \quad dv = 2x\,dx$

$$= \int \frac{\cancel{x}}{v} \cdot \frac{dv}{2\cancel{x}}$$

$$= \frac{1}{2} \ln|x^2+4| - \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right)$$

$$= \ln|x| + \frac{1}{2} \ln(x^2+4) - \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + K \quad \#$$

Ex. $\displaystyle\int \frac{4x^2-3x+2}{4x^2-4x+3}$

$\downarrow$

Degree not less than, thus can divide

$$= \begin{array}{r|l} & \quad 1 \\ 4x^2-4x+3 & \overline{4x^2-3x+2} \\ & 4x^2-4x+3 \\ & \quad x-1 \end{array}$$

$$1 + \frac{x-1}{4x^2-4x+3}$$

$$= \int 1 + \frac{x-1}{(4x^2-4x+3)}$$

$\downarrow$

$$\boxed{4x^2-4x+3 =} \quad (2x-1)^2 + 2$$

ถามพี่ปอน

ต้อง แยก สองตัวแรก
ให้ได้เหมือนเป๊ะ

$v = 2x - 1 \qquad dv = 2\,dx \rightarrow dx = \frac{dv}{2}$

$$= \int 1 + \frac{x-1}{(2x-1)^2+2}$$

$$v = 2x - 1 \qquad \frac{v+1}{2} = \frac{2x}{2} \qquad \frac{v+1}{2} = x$$

$$= x + \int \frac{\frac{v+1}{2} - 1}{(v)^2 + 2} \frac{dv}{2}$$ → อย่าลืม เปลี่ยน dx เป็น dv.

$$= x + \frac{1}{2}\int \frac{\frac{v+1-2}{2}}{v^2+2} \longrightarrow$$

$$= x + \frac{1}{4}\int \frac{v-1}{v^2+2}$$

$$x + \frac{1}{4}\left[\int \frac{v}{v^2+2} - \int \frac{1}{v^2+2}\right.$$

$$\downarrow$$

$$V = v^2 + 2$$

$$dV = 2v\,dv$$

$$dv = \frac{dV}{2v}$$

$$= \int \frac{\cancel{v}}{V} \cdot \frac{dV}{2\cancel{v}}$$

$$= x + \frac{1}{4}\left[\frac{1}{2}\ln|v^2+2| - \frac{1}{\sqrt{2}} \cdot \tan^{-1}\left(\frac{v}{\sqrt{2}}\right)\right.$$

$$= x + \frac{1}{8}\ln|\underset{\overset{\downarrow}{2x-1}}{v}^2+2| - \frac{1}{4\sqrt{2}} \cdot \tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + c \quad //$$

Case 4 : If the denominators contains $(ax^2+bx+c)^r$, where $b^2-4ac<0$, then the partial fraction decomposition contains.

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots, \frac{A_rx+B_r}{(ax^2+bx+c)^r}$$

form of partial fraction decomposition of

$$\frac{x^3+x^2+1}{x(x-1)(x^2+x+1)(x^2+1)^3}$$

$$= \frac{A}{x} + \frac{B}{(x-1)} + \frac{Cx+d}{x^2+x+1} + \frac{Ex+F}{(x^2+1)} + \frac{Gx+H}{(x^2+1)^2} + \frac{Ix+J}{(x^2+1)^3}$$

Example

$$(x^2+1)(x^2+1)$$

$$x^4+x^2+x^2+1$$

$$\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx$$

$$= \left[\frac{1-x+2x^2-x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}\right] x(x^2+1)^2$$

$$= 1-x+2x^2-x^3 = A(x^2+1)^2 + Bx+C((x)(x^2+1)) + (Dx+E)(x)$$

$$= \cancel{Ax^4} + \cancel{2Ax^2} + A + \cancel{bx^4} + \cancel{bx^2} + \cancel{Cx^3} + \cancel{Cx} + \cancel{Dx^2} + \cancel{Ex}$$

$$1+B=0$$

$$= x^4(A+B) + Cx^3 + x^2(2A+b+D) + (E+C)x + A$$

$$A=1, \quad B=-1, \quad C=-1, \quad D=1, \quad E=0$$

$$= \int \left(\frac{1}{x} - \frac{x+1}{x^2+1} + \frac{x}{(x^2+1)^2} \right) dx$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+1) - \tan^{-1}x - \frac{1}{2(x^2+1)} + K$$ #

Ex. 2

$$= \int \frac{\sqrt{x+4}}{x}\, dx$$

$v^2 = \sqrt{x+4}^{\,2}$

$v^2 = x+4$

$v^2 - 4 = x$

$v = \sqrt{x+4}$

$dv = \frac{1}{2}(x+4)^{-\frac{1}{2}}\, dx$

$dv = \frac{1}{2\sqrt{x+4}}\, dx$

$dx = dv\, 2\sqrt{x+4}$

$$= \int \frac{\sqrt{v^2-4+4}}{v^2-4}\, dx$$

$$= \int \frac{v}{v^2-4} \cdot 2v\, dv$$

$$= \int \frac{2v^2}{v^2-4}$$

$$= 2\int \frac{v^2}{v^2-4}$$

↓

$$= 2\int \frac{v^2}{(v+2)(v-2)} = \frac{A}{v+2} + \frac{B}{v-2}$$

Partial fraction ใช้ไม่ได้

can only be use if degree in numerator less than denominator.

$\frac{P(x)}{Q(x)} = P(x) < Q(x)$

$v^2 = A(v-2) + B(v+2)$

$v = 2$ | $v = -2$

$4 = 4B \quad B = 1$ | $4 = -4A$

$A = -1$

↓

$$= 2\int \frac{1}{v-2} - \frac{1}{v+2}$$

$$\frac{1}{v-2}$$

$$= 2\left[\ln|v-2| - \ln|v+2|\right]$$

$$= 2\left[\ln\left|\frac{v-2}{v+2}\right|\right]$$

$$= 2\ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right| + C \quad \#$$

X

ผิด

New ways

$$2\int \frac{v^2}{v^2-4}$$

$$\begin{array}{r|l} & 1 \\ v^2-4 & v^2 \\ & v^2-4 \\ & \quad 4 \end{array}$$

$$= 2\int 1 + \frac{4}{v^2-4}$$

$$= 2\left[\int 1 + \int \frac{4}{v^2-4}\right.$$

$$\downarrow$$

$$2\left[v + 4\int \frac{1}{v^2-4}\right. \nearrow$$

ใช้ไม่ได้ $\frac{1}{x^2+a^2}$

$$2\left[v + \int \frac{4}{(v+2)(v-2)}\right. \longrightarrow \quad 4 = A(v-2) + B(v+2)$$

$$B = 1 \qquad A = -1$$

$$\downarrow$$

$$= \int \frac{1}{v-2} - \frac{1}{v+2}$$

$$= \ln|v-2| - \ln|v+2|$$

$$= 2\left[v + \ln\left|\frac{v-2}{v+2}\right|\right]$$

$$= 2v + 2\ln\left|\frac{v-2}{v+2}\right| + c$$

$$= 2\sqrt{x+4} + 2\ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right| + c \quad ✓$$

Useful tricks !!!

$$= \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C \quad ✓$$

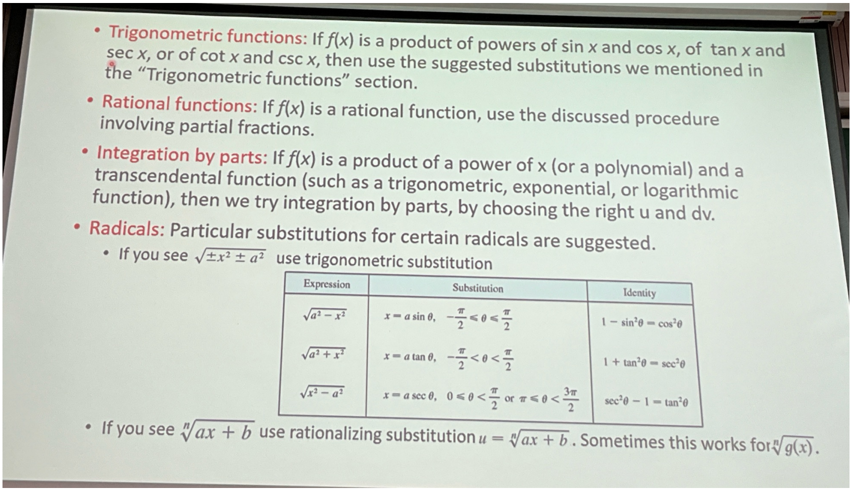
Integration Strategy

Step 1: Simplify the integrand.

$$\int \sqrt{x}\,(1+\sqrt{x})\,dx = \int (\sqrt{x} + x)\,dx$$

Step 2: look for substitution

Step 3: Classify base on the form.

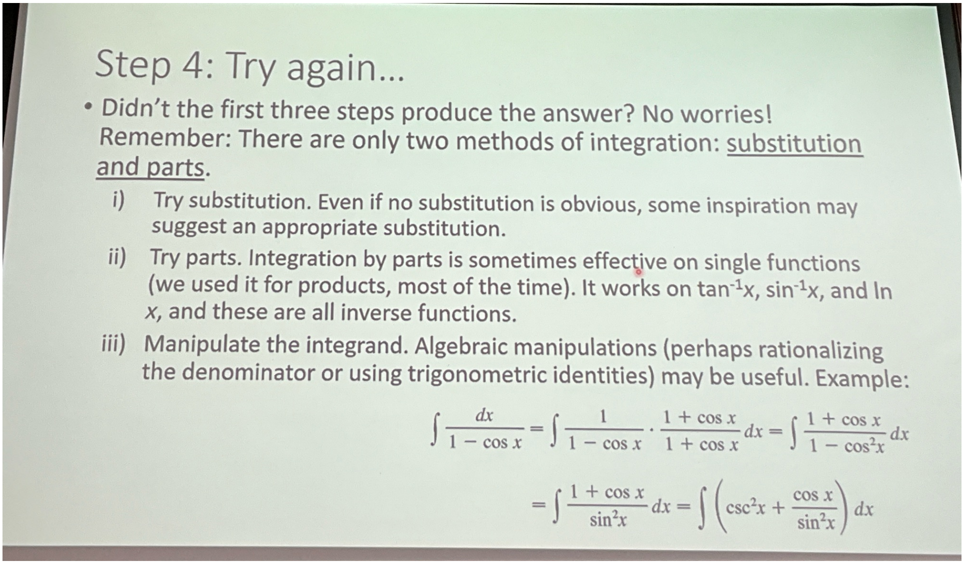


- **Trigonometric functions:** If $f(x)$ is a product of powers of sin x and cos x, of tan x and sec x, or of cot x and csc x, then use the suggested substitutions we mentioned in the "Trigonometric functions" section.
- **Rational functions:** If $f(x)$ is a rational function, use the discussed procedure involving partial fractions.
- **Integration by parts:** If $f(x)$ is a product of a power of x (or a polynomial) and a transcendental function (such as a trigonometric, exponential, or logarithmic function), then we try integration by parts, by choosing the right u and dv.
- **Radicals:** Particular substitutions for certain radicals are suggested.
  - If you see $\sqrt{\pm x^2 \pm a^2}$ use trigonometric substitution

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta,\quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta,\quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta,\quad 0 \leqslant \theta < \frac{\pi}{2}$ or $\pi \leqslant \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

- If you see $\sqrt[n]{ax+b}$ use rationalizing substitution $u = \sqrt[n]{ax+b}$. Sometimes this works for $\sqrt[n]{g(x)}$.

Step 4:



Step 4: Try again…

- Didn't the first three steps produce the answer? No worries! Remember: There are only two methods of integration: <u>substitution and parts</u>.
  - i) Try substitution. Even if no substitution is obvious, some inspiration may suggest an appropriate substitution.
  - ii) Try parts. Integration by parts is sometimes effective on single functions (we used it for products, most of the time). It works on $\tan^{-1}x$, $\sin^{-1}x$, and ln *x*, and these are all inverse functions.
  - iii) Manipulate the integrand. Algebraic manipulations (perhaps rationalizing the denominator or using trigonometric identities) may be useful. Example:

$$\int \frac{dx}{1-\cos x} = \int \frac{1}{1-\cos x} \cdot \frac{1+\cos x}{1+\cos x}\,dx = \int \frac{1+\cos x}{1-\cos^2 x}\,dx$$

$$= \int \frac{1+\cos x}{\sin^2 x}\,dx = \int \left(\csc^2 x + \frac{\cos x}{\sin^2 x}\right) dx$$

Ex. $\int \frac{\tan^3 x}{\cos^3 x}\, dx$

$= \int \frac{\sin^3 x}{\cos^3 x} \cdot \frac{1}{\cos^3 x}\, dx$

$= \int \frac{\sin^3 x}{\cos^6 x}\, dx$

$= \int \frac{(\sin^2 x)(\sin x)}{\cos^6 x}\, dx$

$= \int \frac{(1-\cos^2 x)(\sin x)}{\cos^6 x}\, dx$

$v = \cos x \qquad dv = -\sin x\, dx$

$dx = \frac{dv}{-\sin x}$

$= \int \frac{(1-v^2)}{v^6} \cdot \cancel{\sin x} \cdot \frac{dv}{-\cancel{\sin x}}$

$= -\int \frac{(1-v^2)}{v^6}\, dv$

$= -\left[\int v^{-6} - \int v^{-4}\right.$

$= -\left[\frac{v^{-5}}{-5} - \frac{v^{-3}}{-3}\right.$

$= \frac{1}{5v^5} - \frac{1}{3v^3} = \frac{1}{5\cos^5 x} - \frac{1}{3\cos^3 x} + C$ #

Ex.2 $\int e^{\sqrt{x}}\,dx$

$u^2 = \sqrt{x}^2$

$u^2 = x$

$du = \frac{1}{2} x^{-\frac{1}{2}}\,dx$

$du = \frac{1}{2\sqrt{x}}\,dx$

$dx = 2\sqrt{x}\,du$

$= \int e^u \cdot 2u\,du$

$= 2\int u e^u\,du$

by part

change to $\int x e^x\,dx$

$u = x \quad du = dx$

$dv = e^x\,dx \quad v = e^x$

$= xe^x - \int e^x\,dx$

$= xe^x - e^x$

$= 2\left[ue^u - e^u\right]$

$= 2\left[\sqrt{x}e^{\sqrt{x}} - e^{\sqrt{x}}\right]$

$= 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$ #

Ex.3 $\int \frac{x^5+1}{x^3-3x^2-10x}\,dx$

$$\begin{array}{r|l} & x^2+3x+19 \\ x^3-3x^2-10x & x^5 \; 0 \; 0 \; 0 \; 0 \; 1 \end{array}$$

$x^5-3x^4-10x^3$

$3x^4+10x^3+0$

$3x^4-9x^3-30x^2$

$19x^3+30x^2+0$

$19x^3-57x^2-190x+1$

$87x^2+190x+1$

$= \int x^3+3x+19 + \frac{87x^2+190x+1}{x^3-3x^2-10x}$

$\frac{87x^2+190x+1}{x(x^2-3x-10)} = \frac{A}{x} + \frac{B}{(x-5)} + \frac{C}{(x+2)}$

$(x-5)(x+2)$

$: 87x^2+190x+1 = A(x-5)(x+2) + Bx(x+2) + C(x-5)x$

$= \cancel{A(x^2}-3x-10) + \cancel{Bx^2+2Bx} + \cancel{Cx^2-5Cx}$

$= x^2(A+B+C) + x(-3A+2B-5C) - 10A$

$-10A = 1$

$A = -\frac{1}{10}$

$A+B+C = 87$

$-3A+2B-5C = 190$

$B+C = \frac{871}{10}$

$2B-5C = \frac{1897}{10}$

$B = \frac{3726}{35}$, $C = -\frac{31}{14}$

Ex.4 $\int \frac{dx}{x\sqrt{\ln x}}$

$v = \ln x \quad dv = \frac{1}{x}\,dx$

$dx = x\,dv$

$= \int \frac{1}{\cancel{x}\sqrt{v}} \cdot \cancel{x}\,dv$

$= \int v^{-\frac{1}{2}}\,dv$

$= 2\sqrt{\ln x} + c$

$\cancel{c}$

$$\int x^3 + 3x + 19 - \frac{31}{14(x+2)} + \frac{3126}{35(x-5)} - \frac{1}{10x}$$

and find the integration

and Integrate to set answers

Exs. $\int \sqrt{\frac{1-x}{1+x}}\,dx$

$\downarrow$ *

$= \int \frac{\sqrt{1-x}}{\sqrt{1+x}} \cdot \frac{\sqrt{1-x}}{\sqrt{1-x}}\,dx$

$= \int \frac{1-x}{\sqrt{1-x^2}}\,dx$

$= \int \frac{1}{\sqrt{1-x^2}}$ * $- \int \frac{x}{\sqrt{1-x^2}}\,dx$

$\downarrow$

$x = \sin\theta$

don't forget dx

$dx = \cos\theta\,d\theta$

$= \int \frac{1}{\sqrt{1-\sin^2\theta}} \cos\theta\,d\theta$

$= \int \frac{1}{\cos\theta}\,d\theta$

$\downarrow$

$= \int 1\,d\theta$

$\theta = \sin^{-1} x$

$\int \frac{x}{\sqrt{1-x^2}}\,dx$

$v = 1-x^2 \quad dv = -2x\,dx$

$dx = \frac{dv}{-2x}$

$= \int \frac{\cancel{x}}{v} \cdot \frac{dv}{-2\cancel{x}}$

$= -\frac{1}{2}\int \frac{1}{\sqrt{v}}$

$> \sqrt{v}$

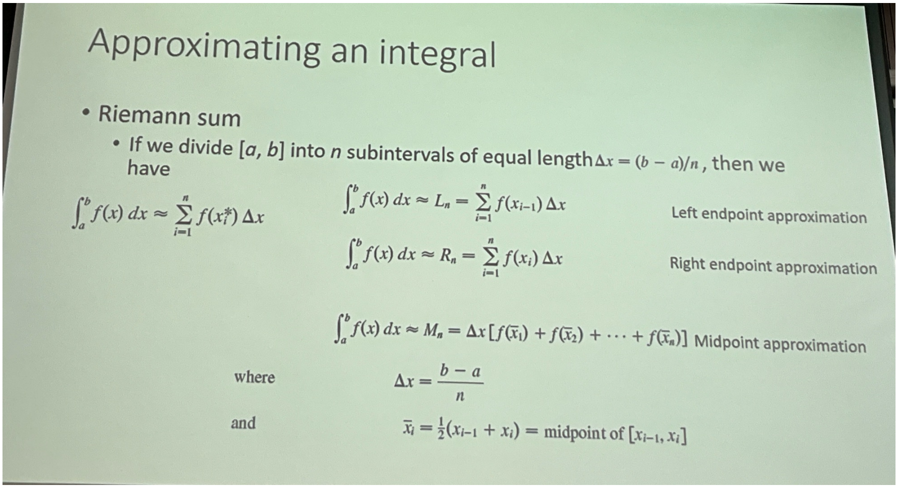
$= \sin^{-1} x - \sqrt{1-x^2} + c$

$\cancel{c}$

How to integrate

$$\int_0^1 e^{x^2}\,dx \qquad \int_{-1}^{1} \sqrt{1+x^3}\,dx$$

Cannot Integrate



Approximating an integral

- Riemann sum
  - If we divide [*a*, *b*] into *n* subintervals of equal length $\Delta x = (b-a)/n$, then we have

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

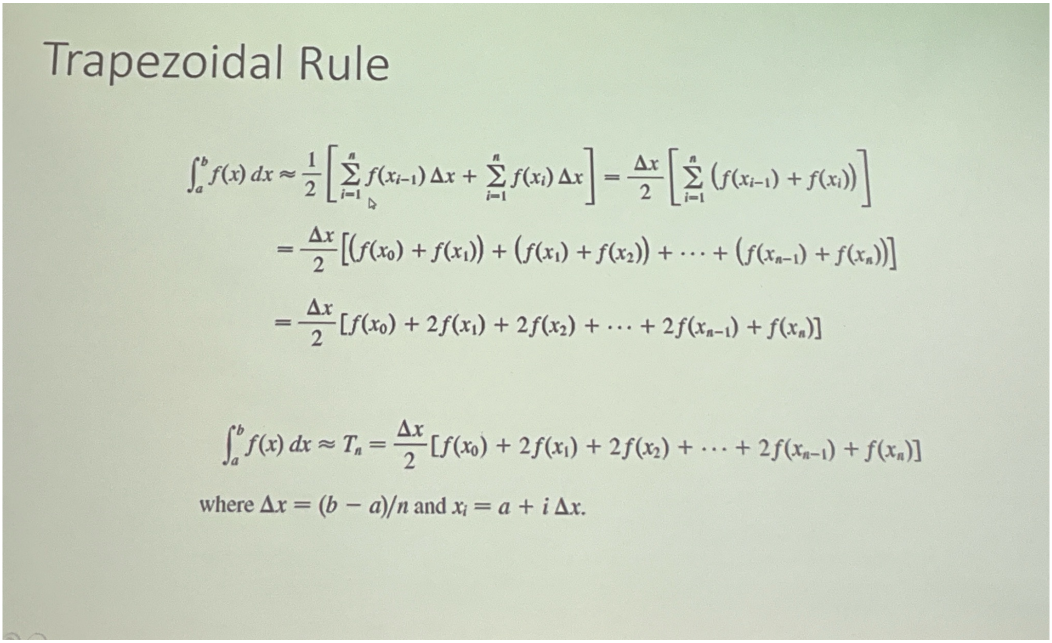
$$\int_a^b f(x)\,dx \approx L_n = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$ Left endpoint approximation

$$\int_a^b f(x)\,dx \approx R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x$$ Right endpoint approximation

$$\int_a^b f(x)\,dx \approx M_n = \Delta x\,[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)]$$ Midpoint approximation

where $$\Delta x = \frac{b-a}{n}$$

and $$\bar{x}_i = \tfrac{1}{2}(x_{i-1} + x_i) = \text{midpoint of } [x_{i-1}, x_i]$$



Trapezoidal Rule

$$\int_a^b f(x)\,dx \approx \frac{1}{2}\left[\sum_{i=1}^{n} f(x_{i-1})\,\Delta x + \sum_{i=1}^{n} f(x_i)\,\Delta x\right] = \frac{\Delta x}{2}\left[\sum_{i=1}^{n}\big(f(x_{i-1}) + f(x_i)\big)\right]$$

$$= \frac{\Delta x}{2}\left[\big(f(x_0) + f(x_1)\big) + \big(f(x_1) + f(x_2)\big) + \cdots + \big(f(x_{n-1}) + f(x_n)\big)\right]$$

$$= \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

$$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

where $\Delta x = (b - a)/n$ and $x_i = a + i\,\Delta x$.

<ins>Approximate</ins>< integral $\int_1^2 \left(\frac{1}{x}\right) dx$ based on (a) Trapezoidal

(b) = midpoint rule n = 5.

$$\Delta x = \frac{2-1}{5} = \frac{1}{5}$$

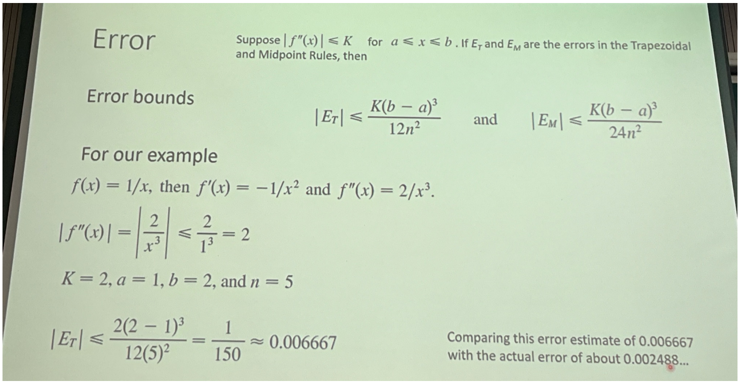
$$= \frac{1}{10}\Big[f(1) + 2f(1.1) + 2f(1.2) + 2f(1.3) + 2f(1.4) + 2f(1.5) + 2f(1.6)$$

$$+ 2f(1.7) + 2f(1.8) + 2f(1.9) + f(2)\Big]$$

$$= \frac{1}{10}\Big[1 + \frac{20}{11} + \frac{5}{3} + \frac{20}{13}$$

Error bounds

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} \quad \text{and} \quad |E_n| \leq \frac{K(b-a)^3}{24n^2}$$



Error

Suppose $|f''(x)| \leqslant K$ for $a \leqslant x \leqslant b$. If E_T and E_M are the errors in the Trapezoidal and Midpoint Rules, then

Error bounds

$$|E_T| \leqslant \frac{K(b-a)^3}{12n^2} \quad \text{and} \quad |E_M| \leqslant \frac{K(b-a)^3}{24n^2}$$

For our example

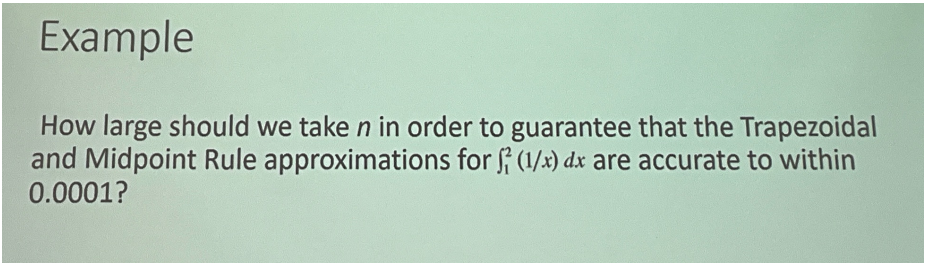
$f(x) = 1/x$, then $f'(x) = -1/x^2$ and $f''(x) = 2/x^3$.

$$|f''(x)| = \left|\frac{2}{x^3}\right| \leqslant \frac{2}{1^3} = 2$$

$K = 2$, $a = 1$, $b = 2$, and $n = 5$

$$|E_T| \leqslant \frac{2(2-1)^3}{12(5)^2} = \frac{1}{150} \approx 0.006667$$

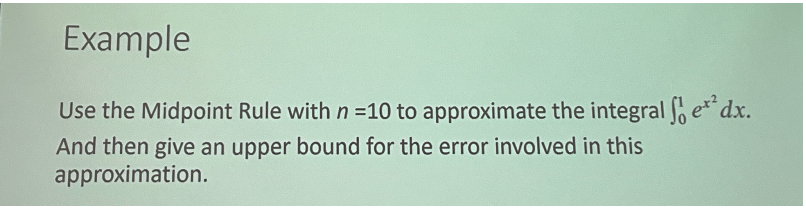
Comparing this error estimate of 0.006667 with the actual error of about 0.002488...



Example

How large should we take *n* in order to guarantee that the Trapezoidal and Midpoint Rule approximations for $\int_1^2 (1/x)\,dx$ are accurate to within 0.0001?

$$[E_T] \leq \frac{K(b-a)^3}{12n^2}$$



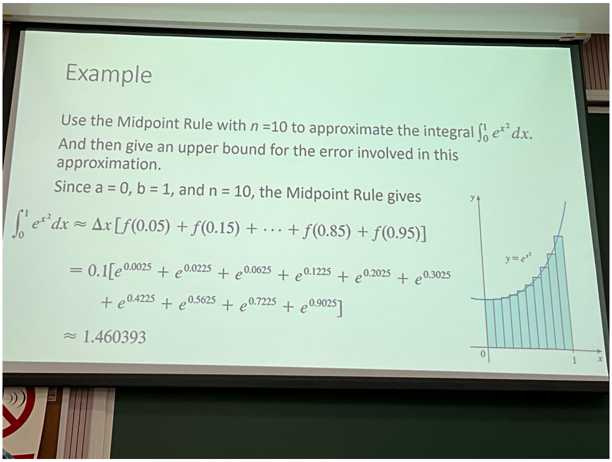
Example

Use the Midpoint Rule with *n* =10 to approximate the integral $\int_0^1 e^{x^2}\,dx$. And then give an upper bound for the error involved in this approximation.

$$\int_0^1 e^{x^2} \qquad \Delta x = \frac{1-0}{10} = \frac{1}{10}$$

$$\cdot \quad \frac{1}{10} \quad \frac{2}{10}$$

$$= \frac{1}{10}\left[f\left(\frac{1.5}{10}\right) + f\left(\frac{2.5}{10}\right) + f\left(\frac{3.5}{10}\right] + f\left(\frac{4.5}{10}\right) + f\left(\frac{5.5}{10}\right)\right.$$



Example

Use the Midpoint Rule with *n* =10 to approximate the integral $\int_0^1 e^{x^2}\,dx$. And then give an upper bound for the error involved in this approximation.

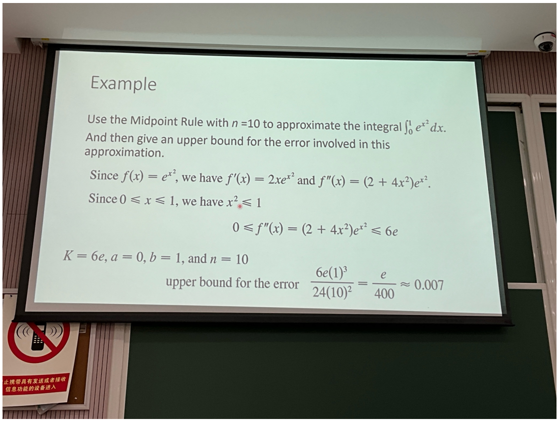
Since a = 0, b = 1, and n = 10, the Midpoint Rule gives

$$\int_0^1 e^{x^2}dx \approx \Delta x\,[f(0.05) + f(0.15) + \cdots + f(0.85) + f(0.95)]$$

$$= 0.1[e^{0.0025} + e^{0.0225} + e^{0.0625} + e^{0.1225} + e^{0.2025} + e^{0.3025}$$

$$+ e^{0.4225} + e^{0.5625} + e^{0.7225} + e^{0.9025}]$$

$$\approx 1.460393$$



Example

Use the Midpoint Rule with *n* =10 to approximate the integral $\int_0^1 e^{x^2}\,dx$. And then give an upper bound for the error involved in this approximation.

Since $f(x) = e^{x^2}$, we have $f'(x) = 2xe^{x^2}$ and $f''(x) = (2 + 4x^2)e^{x^2}$.

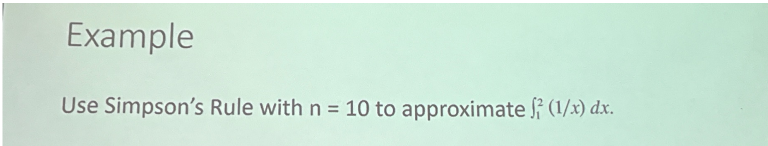
Since $0 \leqslant x \leqslant 1$, we have $x^2 \leqslant 1$

$$0 \leqslant f''(x) = (2 + 4x^2)e^{x^2} \leqslant 6e$$

$K = 6e$, $a = 0$, $b = 1$, and $n = 10$

upper bound for the error $\dfrac{6e(1)^3}{24(10)^2} = \dfrac{e}{400} \approx 0.007$

Simpson Rule for approximation



Example

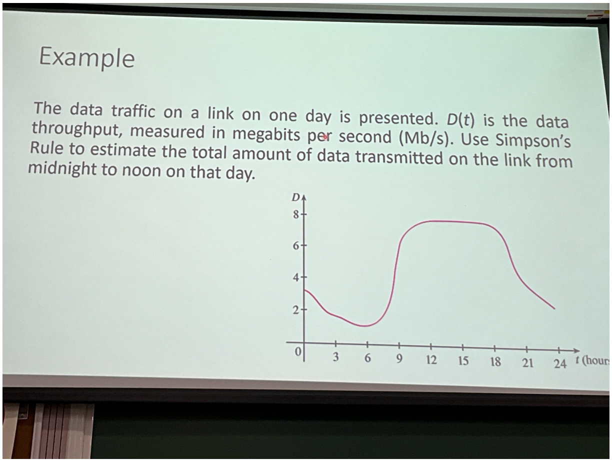
Use Simpson's Rule with n = 10 to approximate $\int_1^2 (1/x)\,dx$.

$$\Delta x = \frac{2-1}{10} = \frac{\frac{1}{10}}{3} = \frac{1}{30}$$

$$= \frac{1}{30}\left[f(1, \right.$$

ln2

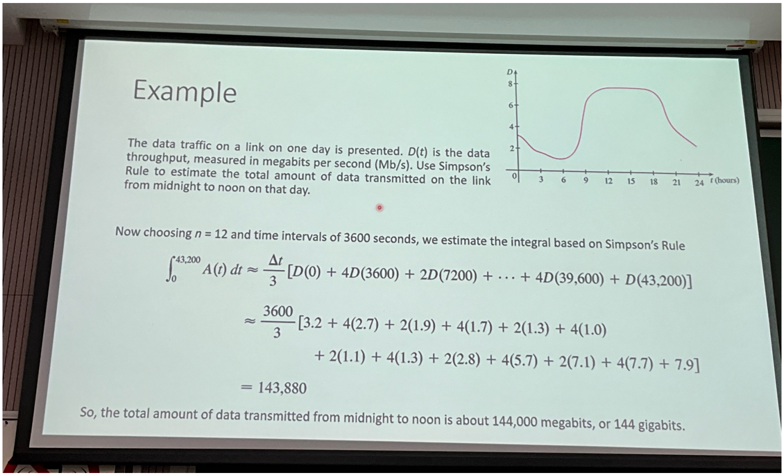
$$S_{2n} = \frac{1}{3}T_n + \frac{2}{3}M_n$$



Example

The data traffic on a link on one day is presented. *D*(*t*) is the data throughput, measured in megabits per second (Mb/s). Use Simpson's Rule to estimate the total amount of data transmitted on the link from midnight to noon on that day.

$$\int_0^{24}$$



Example

The data traffic on a link on one day is presented. $D(t)$ is the data throughput, measured in megabits per second (Mb/s). Use Simpson's Rule to estimate the total amount of data transmitted on the link from midnight to noon on that day.

Now choosing $n = 12$ and time intervals of 3600 seconds, we estimate the integral based on Simpson's Rule

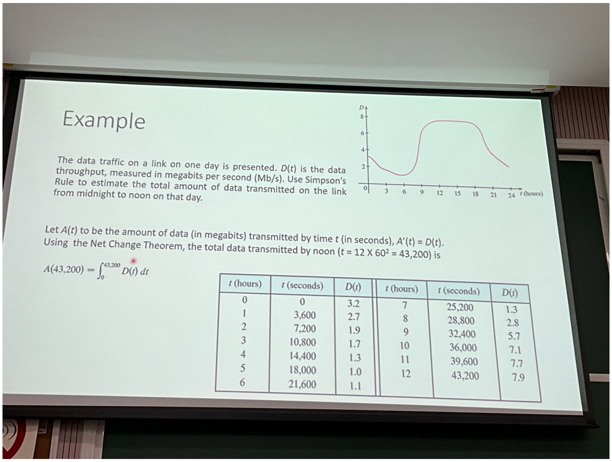
$$\int_0^{43,200} A(t)\,dt \approx \frac{\Delta t}{3}[D(0) + 4D(3600) + 2D(7200) + \cdots + 4D(39,600) + D(43,200)]$$

$$\approx \frac{3600}{3}[3.2 + 4(2.7) + 2(1.9) + 4(1.7) + 2(1.3) + 4(1.0)$$

$$+ 2(1.1) + 4(1.3) + 2(2.8) + 4(5.7) + 2(7.1) + 4(7.7) + 7.9]$$

$$= 143,880$$

So, the total amount of data transmitted from midnight to noon is about 144,000 megabits, or 144 gigabits.



Example

The data traffic on a link on one day is presented. $D(t)$ is the data throughput, measured in megabits per second (Mb/s). Use Simpson's Rule to estimate the total amount of data transmitted on the link from midnight to noon on that day.

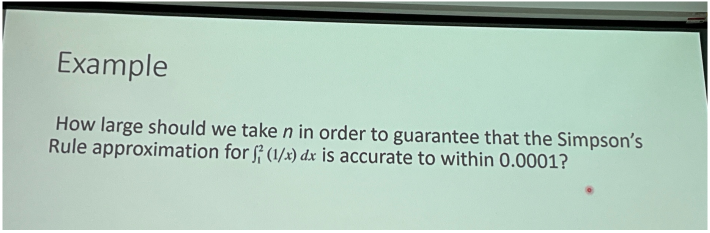
Let $A(t)$ to be the amount of data (in megabits) transmitted by time t (in seconds), $A'(t) = D(t)$. Using the Net Change Theorem, the total data transmitted by noon ($t = 12 \text{ X } 60^2 = 43,200$) is

$$A(43,200) = \int_0^{43,200} D(t)\,dt$$

t (hours)	t (seconds)	$D(t)$	t (hours)	t (seconds)	$D(t)$
0	0	3.2	7	25,200	1.3
1	3,600	2.7	8	28,800	2.8
2	7,200	1.9	9	32,400	5.7
3	10,800	1.7	10	36,000	7.1
4	14,400	1.3	11	39,600	7.7
5	18,000	1.0	12	43,200	7.9
6	21,600	1.1			

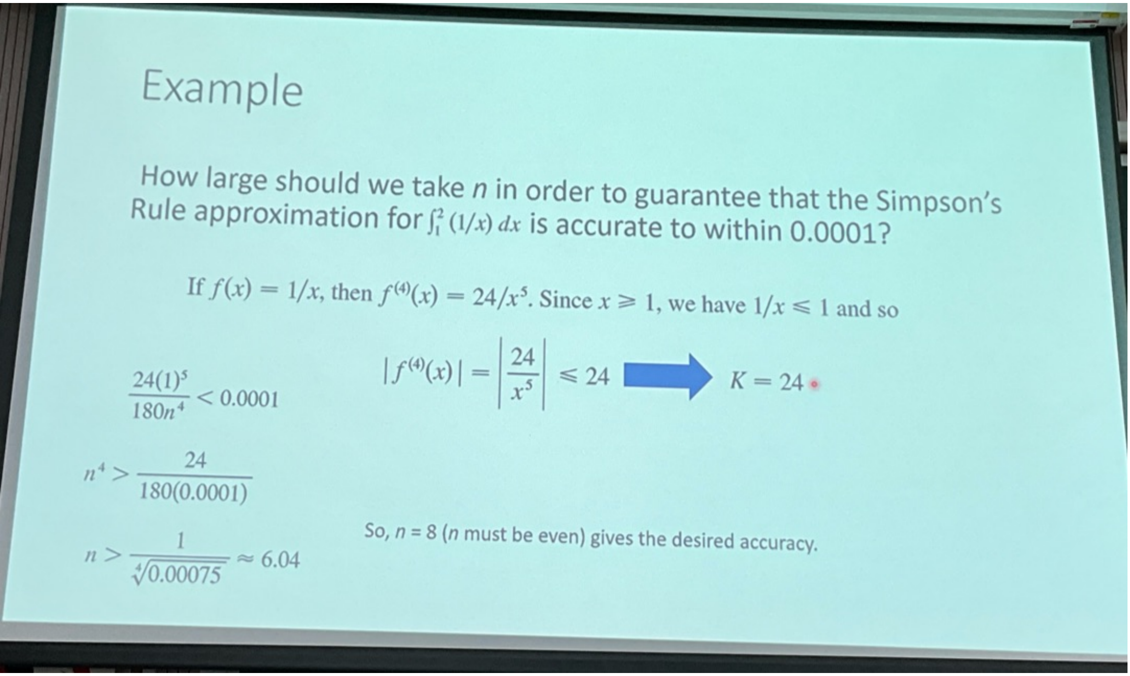
Error bound for simpson $k =$ $|f^{(4)}(x)| \leq k$ for $a \leq x \leq b$

$$|E_s| \leq \frac{K(b-a)^5}{180n^4}$$



Example

How large should we take *n* in order to guarantee that the Simpson's Rule approximation for $\int_1^2 (1/x)\,dx$ is accurate to within 0.0001?



Example

How large should we take *n* in order to guarantee that the Simpson's Rule approximation for $\int_1^2 (1/x)\,dx$ is accurate to within 0.0001?

If $f(x) = 1/x$, then $f^{(4)}(x) = 24/x^5$. Since $x \geqslant 1$, we have $1/x \leqslant 1$ and so

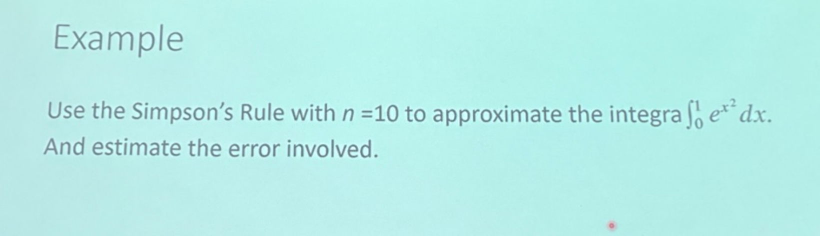
$$|f^{(4)}(x)| = \left|\frac{24}{x^5}\right| \leqslant 24 \Longrightarrow K = 24$$

$$\frac{24(1)^5}{180n^4} < 0.0001$$

$$n^4 > \frac{24}{180(0.0001)}$$

$$n > \frac{1}{\sqrt[4]{0.00075}} \approx 6.04$$

So, *n* = 8 (*n* must be even) gives the desired accuracy.



Example

Use the Simpson's Rule with n =10 to approximate the integra $\int_0^1 e^{x^2}\,dx$. And estimate the error involved.

$$= \quad \Delta x = \frac{1-0}{10} = \frac{1}{10}$$

$$= \frac{1}{30}\left[f(0)+4f\left(\frac{1}{10}\right)+2f\left(\frac{2}{10}\right)+4f\left(\frac{3}{10}\right)+2f\left(\frac{4}{10}\right)+4f\left(\frac{5}{10}\right)+2f\left(\frac{6}{10}\right)+4f\left(\frac{7}{10}\right)\right.$$

$$\left.+2f\left(\frac{8}{10}\right)+4f\left(\frac{9}{10}\right)+f(1)\right]$$

$$\approx 1.462681$$

Error Bound

$$= \quad f^{(4)}(x) = (12+48x^2+16x^4)e^{x^2}$$

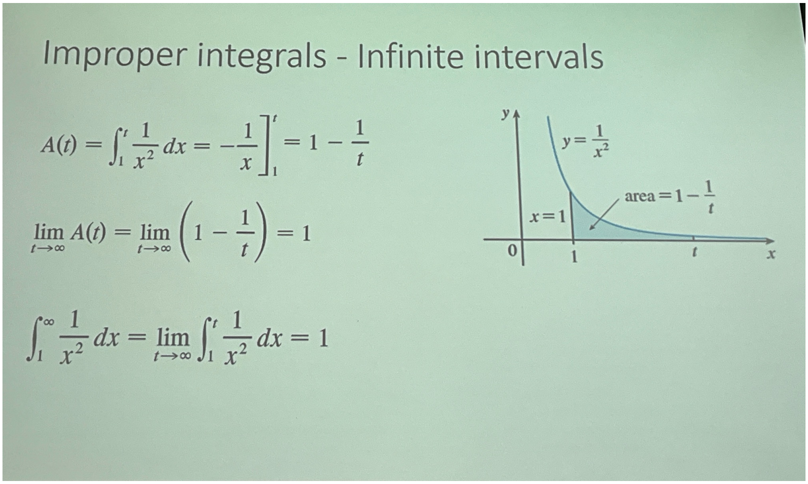
$$0 \le x \le \quad , \quad 0 \le f^{(4)}(x) \le (12+48+16)e^1 = 76e$$

$$= \frac{76e(1)^5}{180(10)^4} \approx 0.000115$$

Mathematica → might be on final

Improper Integrals - Infinite intervals

$\int \sec\theta$

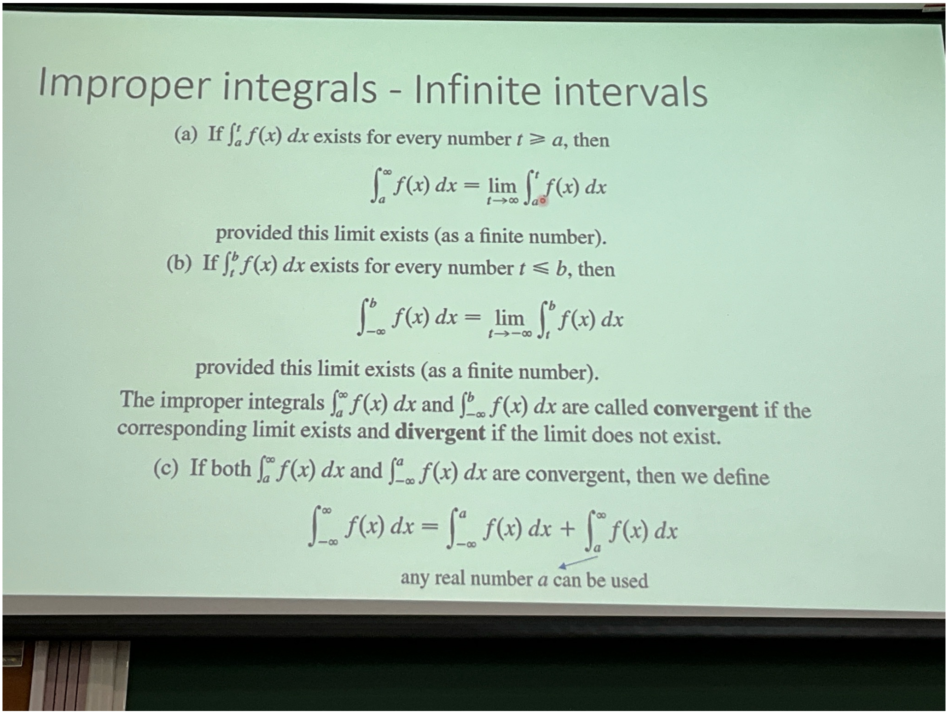


Improper integrals - Infinite intervals

$$A(t) = \int_1^t \frac{1}{x^2}\,dx = -\frac{1}{x}\Big]_1^t = 1 - \frac{1}{t}$$

$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = 1$$

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx = 1$$



Improper integrals - Infinite intervals

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geqslant a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \leqslant b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

any real number a can be used

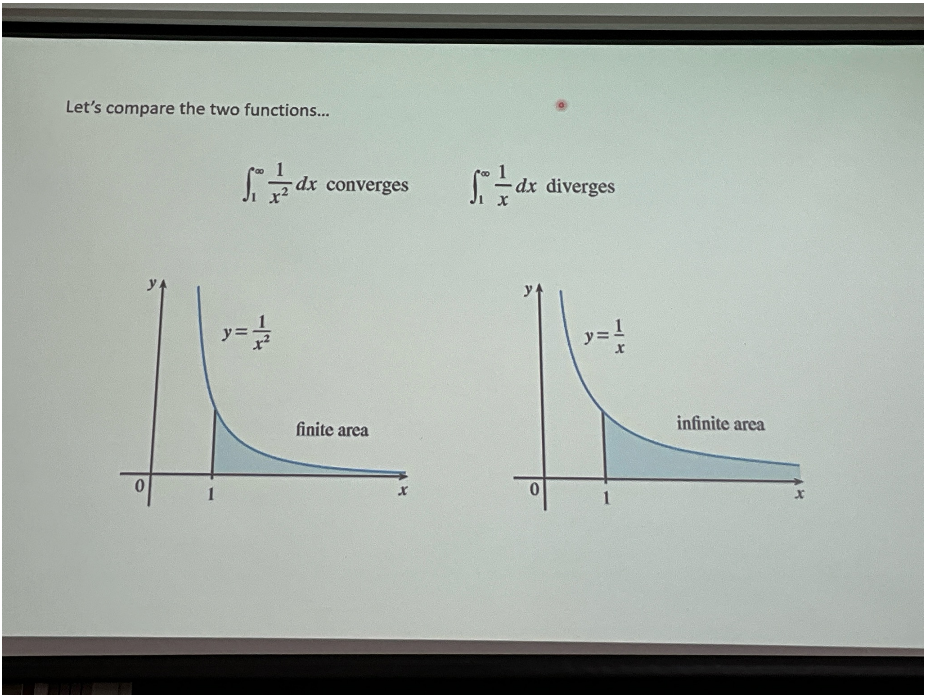
Is this integral converge

$$= \int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$$

$$= \lim_{t\to\infty} \ln(x)\Big|_1^t$$

$$= \lim_{t\to\infty} \ln(t) - \ln(1)$$

$$= \lim_{t\to\infty} \ln(t) = \infty \ \#$$ → Divergent.



Let's compare the two functions...

$\int_1^\infty \frac{1}{x^2}\,dx$ converges

$\int_1^\infty \frac{1}{x}\,dx$ diverges evaluate $\int_{-\infty}^{0} xe^{x}\,dx$

$= \lim_{t \to -\infty} \int_{t}^{0} xe^{x}\,dx$

$v = x$ $\quad dv = dx$

$dv = e^{x}dx$ $\quad v = e^{x}$

$= xe^{x} - \int e^{x}\,dx$

$= xe^{x} - e^{x}$

$= e^{x}(x-1)\Big|_{t}^{0}$

$= e^{0}(0-1) - e^{t}(t-1)$

$= -1 - te^{t} + e^{t}$

$\lim_{t \to -\infty} = -1$ converge

evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

$= \int_{t}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{t} \frac{1}{1+x^2}\,dx$

$= \tan^{-1}(x)\Big|_{t}^{0} + \tan^{-1}(x)\Big|_{0}^{t}$

$= 0 - \tan^{-1}(t)$

$\lim_{t \to -\infty} = 0 - \underset{(-\frac{\pi}{2})}{\tan^{-1}(t)}$

$= 0 + \frac{\pi}{2}$

$* \lim_{t \to +\infty} = \tan^{-1}(t) - 0$

$= \frac{\pi}{2} - 0$

$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$

$\downarrow$

Converge.

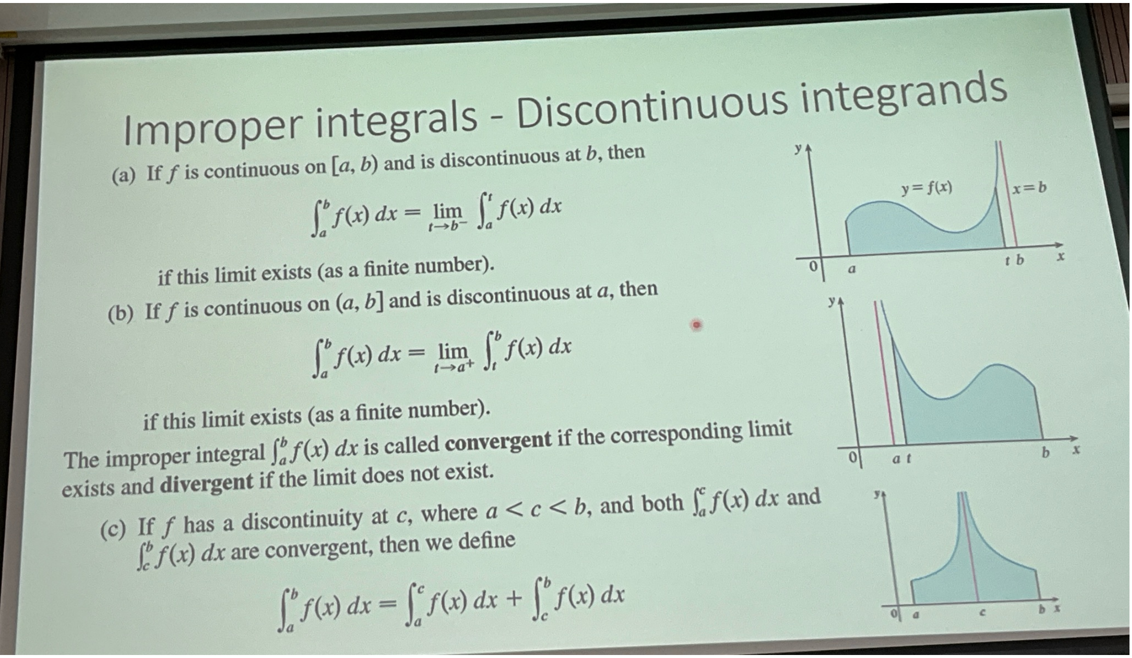
Note: $\arctan(\infty) = \frac{\pi}{2}$

For what values of p is the integral converge

$$\int_1^\infty \frac{1}{x^p} dx$$ convergent $\longrightarrow$ limit exist

$$\int_1^t \frac{1}{x^p} dx$$

$$\int_t^1 \frac{1}{x^p} dx$$



Improper integrals - Discontinuous integrands

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

if this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

if this limit exists (as a finite number).

The improper integral $\int_a^b f(x)\,dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Ex. Find $\int_{2}^{5} \frac{1}{\sqrt{x-2}}\,dx$

dis continuous at 2

$$\int_{2}^{5} \frac{dx}{\sqrt{x-2}} = \lim_{t\to 2^{+}} \int_{t}^{5} \frac{1}{\sqrt{x-2}}\,dx$$

$\int_{0}^{\frac{\pi}{2}} \sec x\,dx$ diverge or converge

dis cont at $\frac{\pi}{2}$

=

$$= \int_{0}^{\frac{\pi}{2}} \sec x\,dx = \lim_{t\to \frac{\pi}{2}^{-}} \int_{0}^{t} \sec x\,dx$$

$$= \lim_{t\to (\frac{\pi}{2})^{-}} \ln|\sec x + \tan x|\Big]_{0}^{t}$$

$\int_0^3 \frac{dx}{x-1}$ if possible .

$$= \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1}$$

$$\int_0^1 \frac{dx}{x-1} = \lim_{t \to 1^-} \int_0^t \frac{dx}{x-1} = \lim_{t \to 1^-} \ln|x-1| \Big]_0^t$$

$$= \lim_{t \to 1^-} (\ln|t-1| - \ln|-1|) = \lim_{t \to 1^-} \ln(1-t) = -\infty$$

$\nexists$

Diverge also imply that $\int_1^3 \frac{dx}{x-1}$

$$\int_0^1 \ln x \; dx$$

discontinue at 0

$$= \lim_{t \to 0^+} \int_t^1 \ln x \, dx \nearrow$$

$u = \ln x \qquad du = \frac{1}{x} dx$

$dv = dx \qquad v = x$

$= x \ln x - x + c$

$$= \lim_{t \to 0^+} x \ln x - x \Big|_t^1$$

$$= \lim_{t \to 0^+} \Big[(\ln(1) - 1) - (t \ln(t) - t) \Big]$$

$$= \lim_{t \to 0^+} \Big[-1 - (t \ln(t) - t) \Big]$$

$$= \lim_{t \to 0^+} \underbrace{-t \ln(t)}_{\text{use l'Hopital's Rule.}} - 1 + t \approx \text{about } -1$$

#

Comparison Theorem

$$\int_1^\infty \frac{1 + e^{-x}}{x} \, dx \longrightarrow \text{as } \frac{1 + e^{-x}}{x} > \frac{1}{x}$$

$\downarrow$

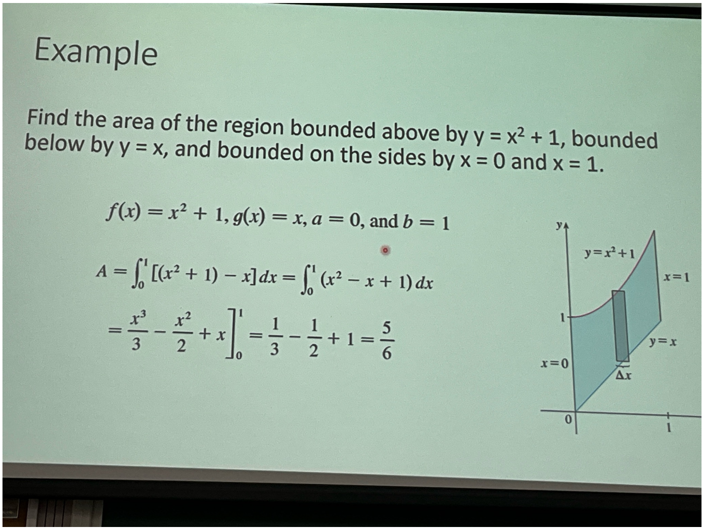
$\int_1^\infty \left(\frac{1}{x} \right) dx$ is divergent so it is also divergent. —> (original)

Fri Mar 15

Area between curve

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)] \Delta x$$

$$A = \int_a^b [f(x) - g(x)]\, dx$$



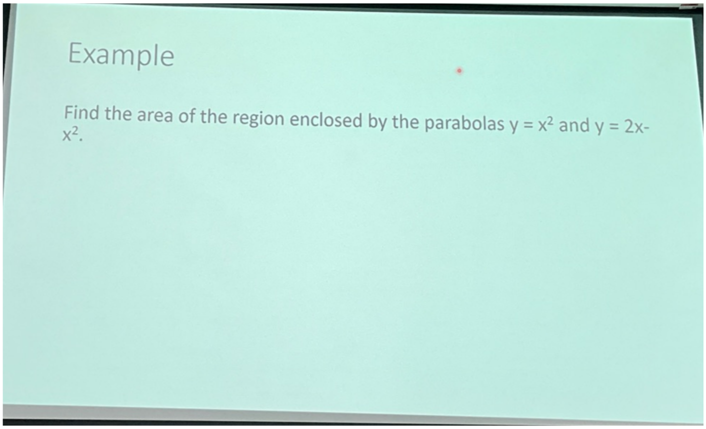
Example

Find the area of the region bounded above by y = x² + 1, bounded below by y = x, and bounded on the sides by x = 0 and x = 1.

$f(x) = x^2 + 1$, $g(x) = x$, $a = 0$, and $b = 1$

$$A = \int_0^1 [(x^2 + 1) - x]\, dx = \int_0^1 (x^2 - x + 1)\, dx$$

$$= \frac{x^3}{3} - \frac{x^2}{2} + x \Big]_0^1 = \frac{1}{3} - \frac{1}{2} + 1 = \frac{5}{6}$$



Example

Find the area of the region enclosed by the parabolas y = x² and y = 2x-x².

$y = x^2 , y = 2x - x^2$

= จับ ค่า side ก่อน for x

= $x^2 = 2x - x^2$

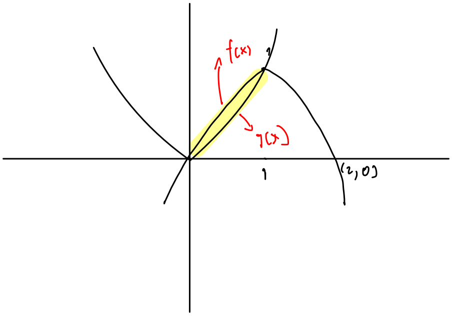
= $\frac{2x^2}{2} = \frac{2x}{2}$

$x^2 = x$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 1, 0$

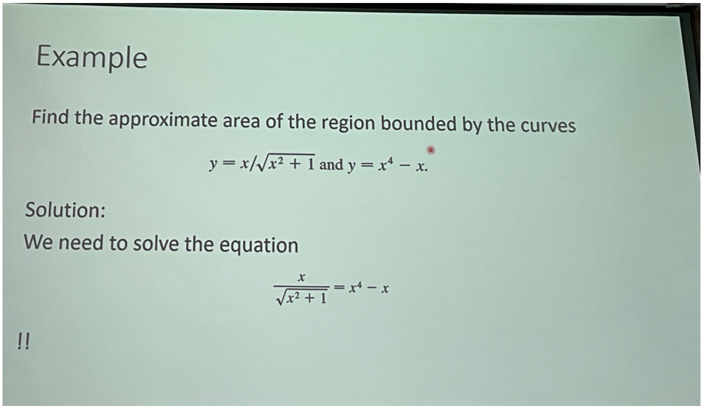


Intersection Point = (0, 0) and (1, 1)

$$= \int_0^1 (2x - x^2) - x^2 = \int_0^1 -2x^2 + 2x \, dx$$

$$= 2\int_0^1 (x - x^2)\, dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}$$



Example

Find the approximate area of the region bounded by the curves

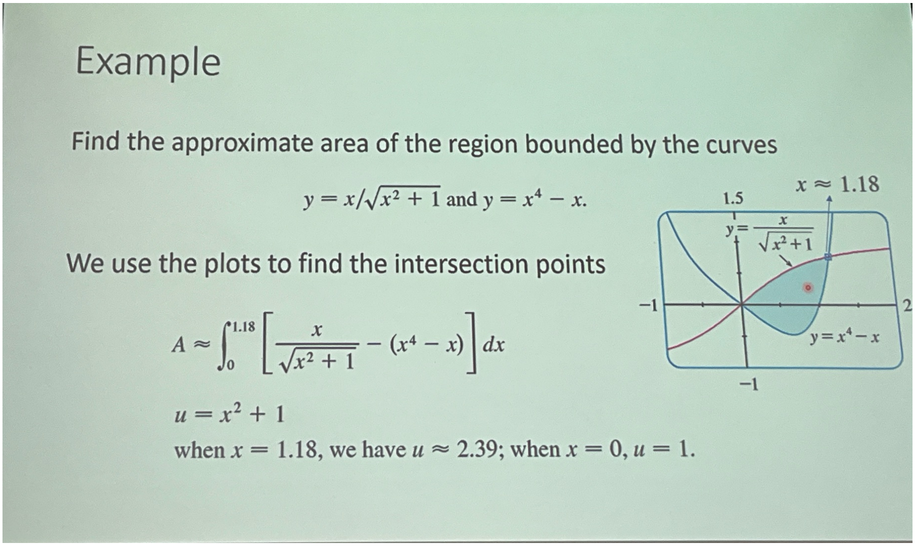
$$y = x/\sqrt{x^2 + 1} \text{ and } y = x^4 - x.$$

Solution:

We need to solve the equation

$$\frac{x}{\sqrt{x^2+1}} = x^4 - x$$

!!



Example

Find the approximate area of the region bounded by the curves

$$y = x/\sqrt{x^2 + 1} \text{ and } y = x^4 - x.$$

We use the plots to find the intersection points

$$A \approx \int_0^{1.18} \left[\frac{x}{\sqrt{x^2+1}} - (x^4 - x) \right] dx$$

$u = x^2 + 1$

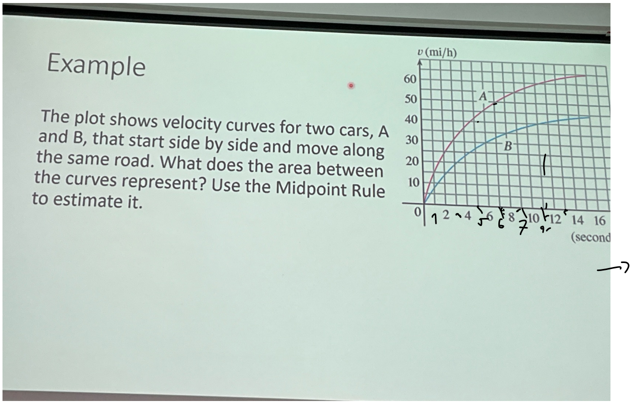
when $x = 1.18$, we have $u \approx 2.39$; when $x = 0$, $u = 1$.

and then find the approximate area,

$$A \approx \int_0^{1.18} \left[\frac{x}{\sqrt{x^2+1}} - (x^4 - x) \right] dx$$

$$= \frac{1}{2} \int_1^{2.39} \frac{du}{\sqrt{u}} - \int_0^{1.18} (x^4 - x)\, dx$$

$$= \sqrt{u} \Big]_1^{2.39} - \left[\frac{x^5}{5} - \frac{x^2}{2} \right]_0^{1.18} \approx 0.785$$



Example

The plot shows velocity curves for two cars, A and B, that start side by side and move along the same road. What does the area between the curves represent? Use the Midpoint Rule to estimate it.

→ Area = difference between distance of the car after 16 second

0 to 16

change to ft/s first

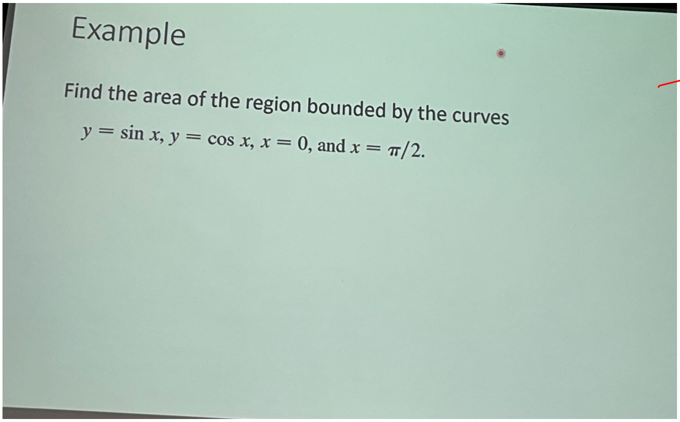
$$\therefore \Delta x = \frac{16-0}{4} = 4$$

$$\therefore \int_a^b (V_a - V_b)\, dt = 4[((V_2) + (V_6) + (V_{10}) + (V_{14})]$$

$$= 4[(14) + (23) + (24) + (30.8)]$$

$$\approx 367.2$$

* $1 \text{ mi/h} = \frac{5280}{3600} \text{ ft/s}$

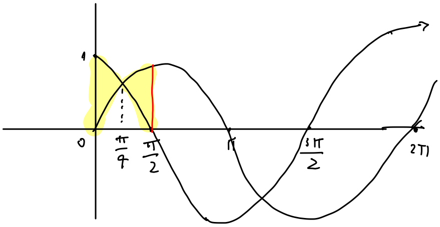


Example

Find the area of the region bounded by the curves

$y = \sin x$, $y = \cos x$, $x = 0$, and $x = \pi/2$.

What if $f(x) \geq g(x)$ for some x but $g(x) \geq f(x)$ for other x.



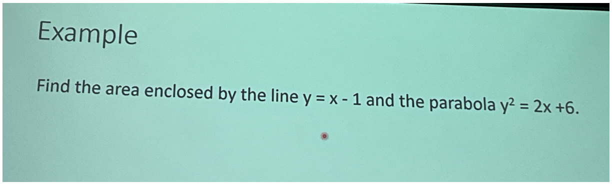
$$\boxed{\sin x = \cos x \text{ at } \frac{\pi}{4}}$$

$$= \int_0^{\frac{\pi}{4}} \cos x - \sin x \; dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x - \cos x \; dx$$

$$= \Big[\sin x + \cos x\Big]_0^{\frac{\pi}{4}} + \Big[-\cos x - \sin x\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

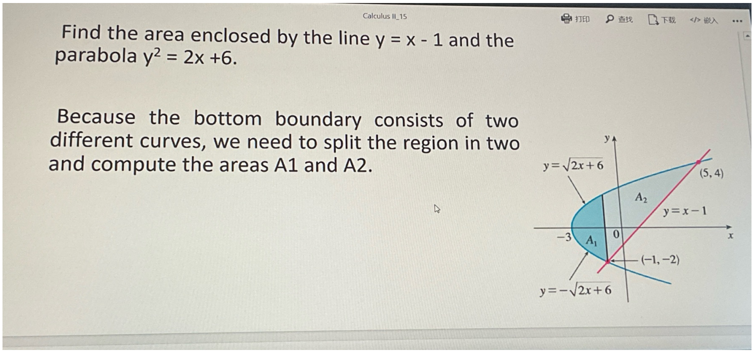
calculate = $2\sqrt{2} - 2$

* Not that clear.



Example

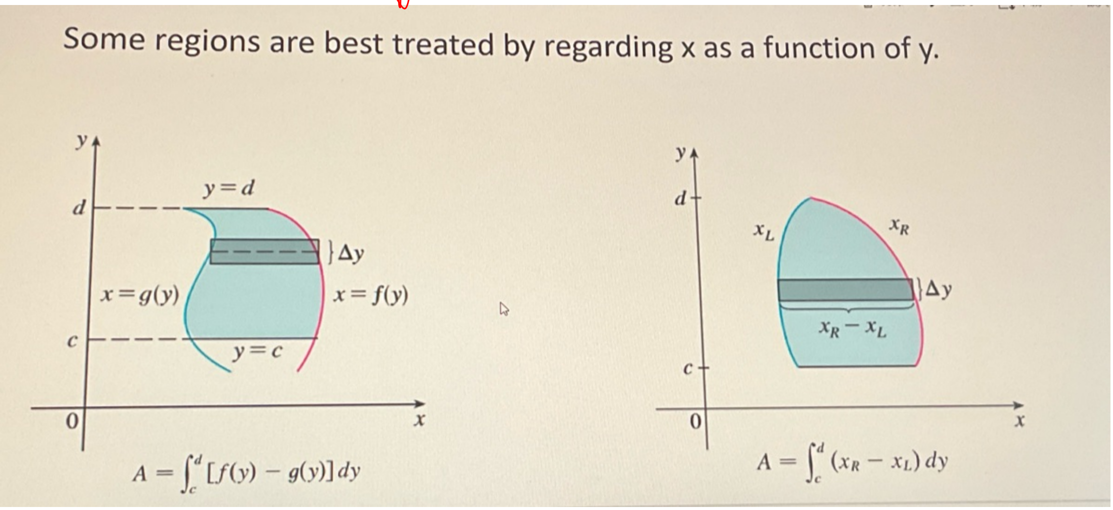
Find the area enclosed by the line y = x - 1 and the parabola $y^2 = 2x + 6$.



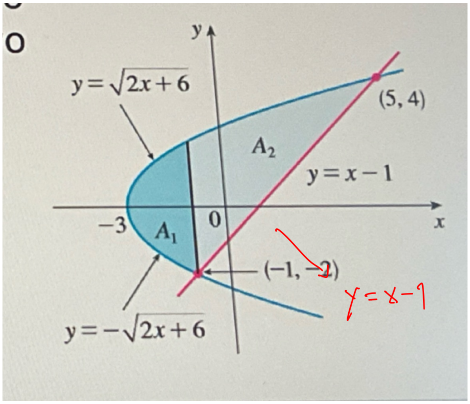
Find the area enclosed by the line y = x - 1 and the parabola $y^2 = 2x + 6$.

Because the bottom boundary consists of two different curves, we need to split the region in two and compute the areas A1 and A2.

Thus, in this case for lower left function



Some regions are best treated by regarding x as a function of y.



find point of intersection between

$y = x-1$ and $y^2 = 2x+6$

$\sqrt{2x+6}^{\,2} = x-1^{\,2}$

$2x+6 = x^2-2x+1$

x^2-4x-5

$= (x-5)(x+1)$

$= x = 5, -1$

For $x=5$ $y=4$ = $(5,4)$

for $x=-1$ $y=-2$ $(-1,-2)$

point of intersection

$y^2 = 2x+6$ ↓ solve parabola equation

$y = x-1$ ↓ solve for x

$\frac{-2x = -y^2+6}{-2}$

$x = y+1$ # → X_R

$x = \frac{y^2}{2} - 3$ # → X_L

y axis point

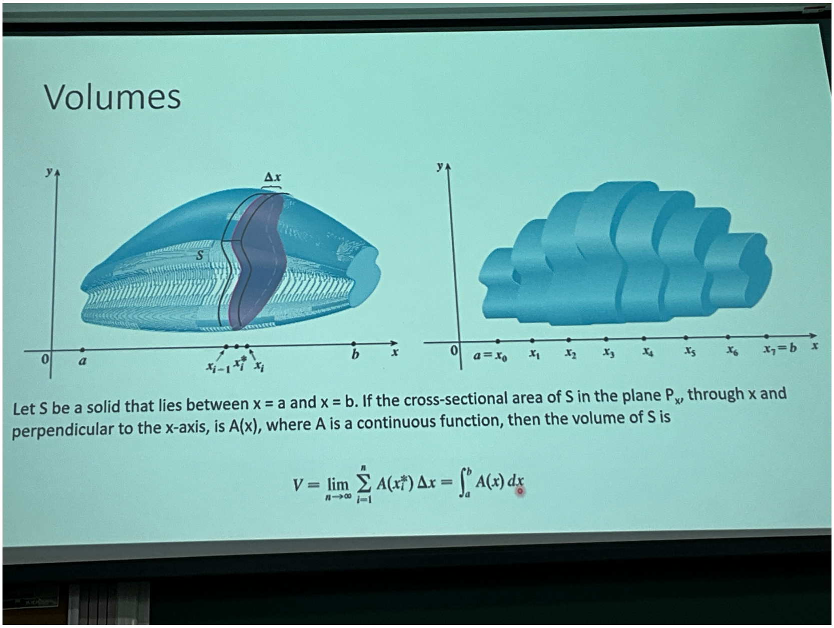
$$A = \int_{-2}^{4} (X_R - X_L)\,dy = \int_{-2}^{4} \left[(y+1) - \left(\frac{1}{2}y^2 - 3\right)\right] dy$$

y axis point

$$= \int_{-2}^{4} \left(-\frac{1}{2}y^2 + y + 4\right) dy = -\frac{1}{2}\left(\frac{y^3}{3}\right) + \frac{y^2}{2} + 4y \Big]_{-2}^{4}$$

$$= \frac{1}{6}(64) + 8 + 16 - \left(\frac{4}{3} + 2 - 8\right) = 18$$ #

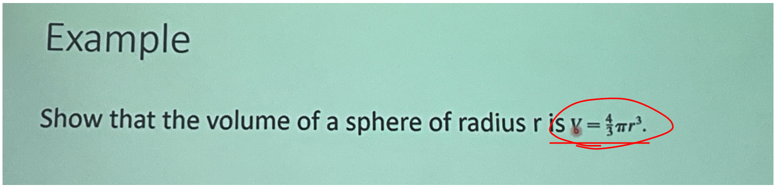
Volumes



Volumes

Let S be a solid that lies between x = a and x = b. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is A(x), where A is a continuous function, then the volume of S is

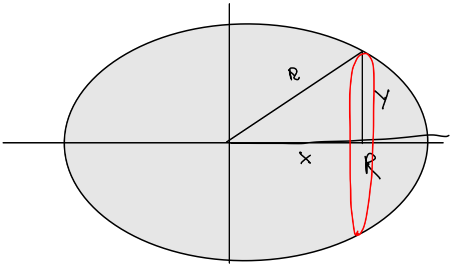
$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\,\Delta x = \int_a^b A(x)\,dx$$



Example

Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\,\Delta x = \int_a^b A(x)\,dx$$



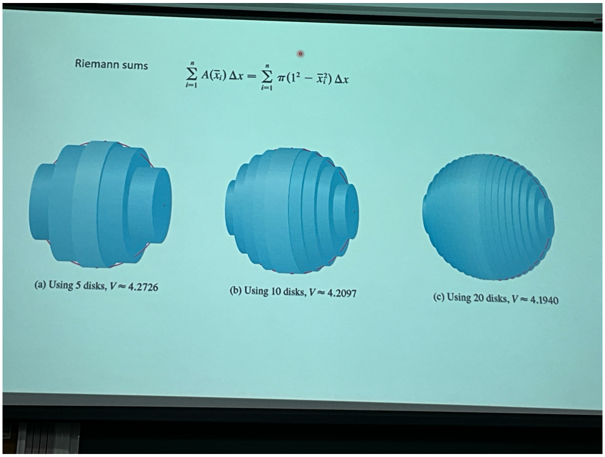
$$x^2 + y^2 = R^2$$

$$\sqrt{y^2} = \sqrt{R^2 - x^2}$$

$$y = \sqrt{R^2 - x^2}$$

$$A = \pi y^2 = \pi\left(\sqrt{r^2 - x^2}\right)^2$$

$$= \int_{-r}^{r} \pi\,(r^2 - x^2)\,dx$$



Riemann sums $\sum_{i=1}^{n} A(\bar{x}_i)\,\Delta x = \sum_{i=1}^{n} \pi(1^2 - \bar{x}_i^2)\,\Delta x$

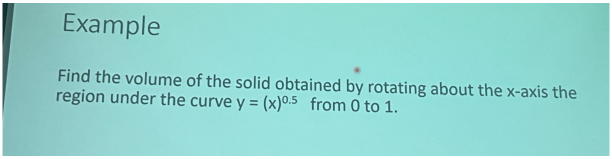
(a) Using 5 disks, $V \approx 4.2726$

(b) Using 10 disks, $V \approx 4.2097$

(c) Using 20 disks, $V \approx 4.1940$

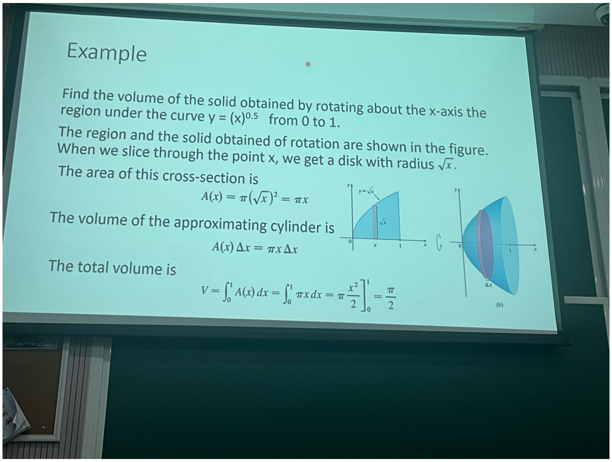
$$= 2\pi \int_0^r r^2 - x^2 \, dx$$

$$= 2\pi \left[r^2 x - \frac{x^3}{3} \right]_0^r = 2\pi \left(r^3 - \frac{r^3}{3} \right)$$



Example

Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = (x)^{0.5}$ from 0 to 1.



Example

Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = (x)^{0.5}$ from 0 to 1.

The region and the solid obtained of rotation are shown in the figure. When we slice through the point x, we get a disk with radius $\sqrt{x}$.

The area of this cross-section is

$$A(x) = \pi(\sqrt{x})^2 = \pi x$$

The volume of the approximating cylinder is

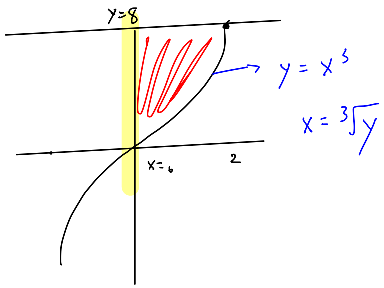
$$A(x)\,\Delta x = \pi x\,\Delta x$$

The total volume is

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi x\,dx = \pi \frac{x^2}{2}\Big]_0^1 = \frac{\pi}{2}$$

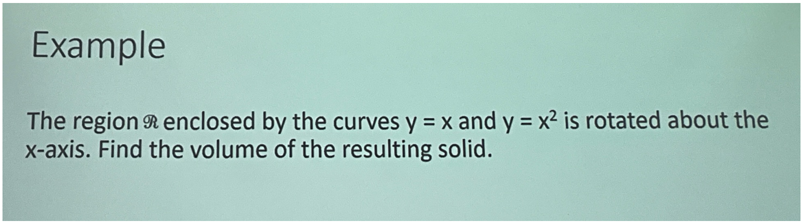
$y = x^3$, $y = 8$ and $x = 0$ about y - axis

y-axis ก็ ดู $(0-8)$



$$A(y) = \pi x^2 = \pi(\sqrt[3]{y})^2 = \pi y^{\frac{2}{3}}$$

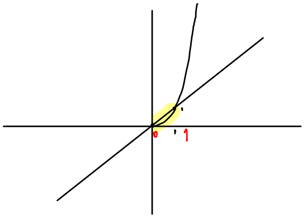
$$\text{volume} = \int_0^8 A(y)\,dy = \int_0^8 \pi y^{\frac{2}{3}}\,dy = \pi\left[\frac{3}{5}y^{\frac{5}{3}}\right]_0^8 = \frac{96\pi}{5}$$



Example

The region ℜ enclosed by the curves y = x and y = x² is rotated about the x-axis. Find the volume of the resulting solid.

$y = x$, $y = x^2$



$x^2 = x$

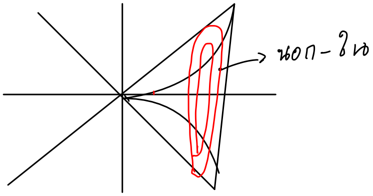
$x^2 - x = 0$

$x(x-1) = 0$

Intersect at $(x = 1, 0)$

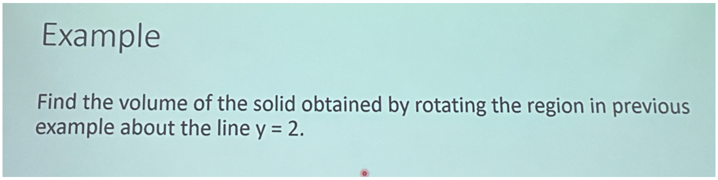
นอก-ใน

$$A(x) = \pi(x)^2 - \pi(x^2)^2$$



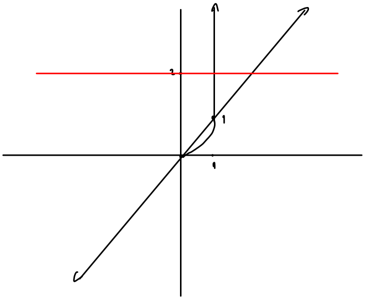
$$= \pi\int_0^1 (x^2 - x^4)\,dx$$

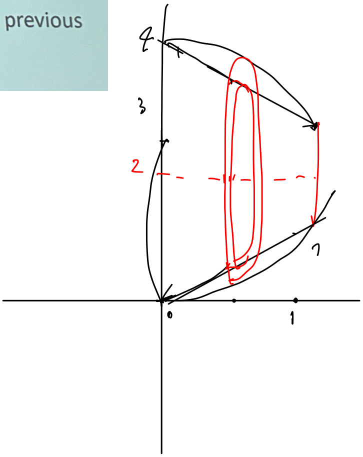
$$= \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{2\pi}{15}$$



Example

Find the volume of the solid obtained by rotating the region in previous example about the line y = 2.





$$A(x) = \pi(2-x^2)^2 - \pi(2-x)^2$$

$$V = \int_0^1 A(x)\,dx$$

$$= \pi \int_0^1 [(2-x^2)^2 - (2-x)^2]\,dx$$

$$= \pi \int_0^1 (x^4 - 5x^2 + 4x)\,dx$$

$$= \pi \left[\frac{x^5}{5} - 5\frac{x^3}{3} + 4\frac{x^2}{2}\right]_0^1$$

$$= \frac{8\pi}{15}$$

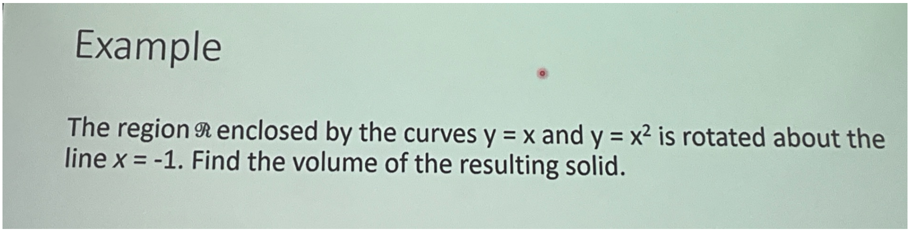
#

Solids of Revolution

$$V = \int_a^b A(x)\,dx \quad \text{or} \quad V = \int_c^d A(y)\,dy$$

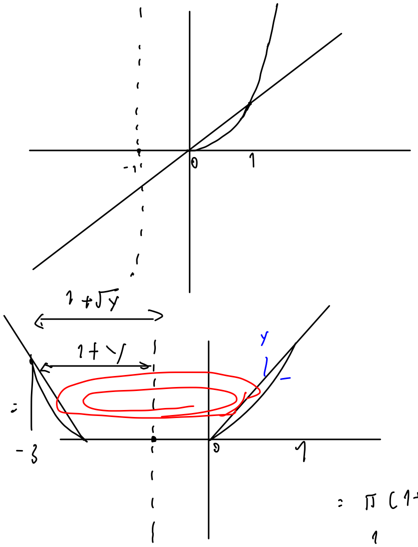
Cross-section area of disk : $A = \pi(\text{radius})^2$

Cross-section area of washer : $A = \pi(\text{outer } R)^2 - \pi(\text{inner } R)^2$.



Example

The region $\mathscr{R}$ enclosed by the curves y = x and y = x² is rotated about the line x = -1. Find the volume of the resulting solid.



In y axis so

!!! Remember to find point of line of intersection first.

In this case

$x = x^2$

$x^2 - x = 0$

$x(x-1) = (x = 0, 1)$ intersect

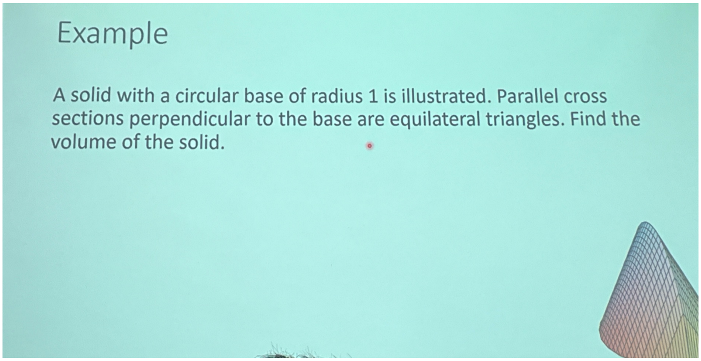
so go from (0, 1)

$= \pi(1+\sqrt{y})^2 - \pi(1+y)^2$

$= \pi\int_0^1 ((1+\sqrt{y})^2 - (1+y)^2)\,dy$

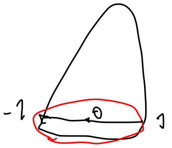
$= \frac{\pi}{2}$

##



Example

A solid with a circular base of radius 1 is illustrated. Parallel cross sections perpendicular to the base are equilateral triangles. Find the volume of the solid.

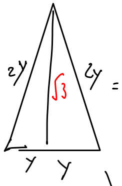


Conic Section

$x^2+y^2=1$

$y^2=1-x^2$

$y=\sqrt{1-x^2}$



= equilateral

$2y^2 = y^2+h^2$

$h=\sqrt{3}\,y$

$$\text{Area} = \frac{1}{2}\cdot 2y\cdot\sqrt{3}\,y$$

$$= \frac{1}{2}\cdot 2\sqrt{1-x^2}\cdot\sqrt{3}\sqrt{1-x^2}$$

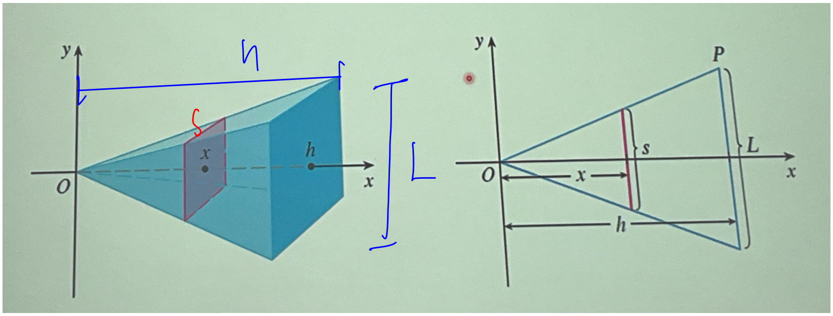
$$A(x) = \sqrt{3}(1-x^2)$$

$$= \int_{-1}^{1}\sqrt{3}\,(1-x^2)\,dx$$

$$= 2\sqrt{3}\int_{0}^{1}(1-x^2)\,dx$$

$$= \frac{4\sqrt{3}}{3}$$

Find the volume of a pyramid whose base is a square with side L and height is h.



$$\frac{s}{x} = \frac{L}{h} \qquad s = \frac{Lx}{h}$$

$$\text{volume} \overset{A}{=} s^2 = \frac{L^2x^2}{h^2}$$

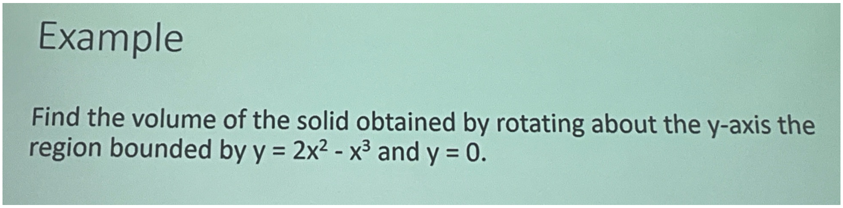
$$= \int_0^h \frac{L^2x^2}{h^2}\,dx$$

$$= \frac{L^2h}{\substack{3 \\ \cancel{h}}}$$

Volumes by cylindrical shells

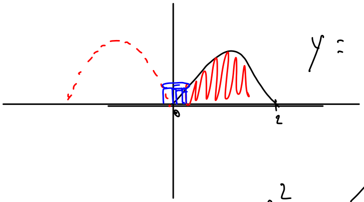
$$V_i = (2\pi \bar{x}_i)[f(\bar{x}_i)]\,\Delta x$$

$$V = \int_a^b 2\pi x\, f(x)\, dx$$ where $0 \le a < b$



Example

Find the volume of the solid obtained by rotating about the y-axis the region bounded by y = 2x² - x³ and y = 0.



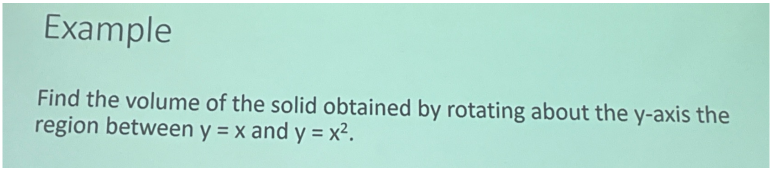
$y = x^2(2-x)$

$x = 2, 0$

$$V = \int_0^2 (2\pi \overset{\text{width.}}{x})(2x^2 - x^3)\, dx$$

$$= 2\pi \int_0^2 (2x^3 - x^4)\, dx = 2\pi \left[\frac{1}{2}x^4 - \frac{1}{5}x^5\right]_0^2$$

$$= 2\pi\left(8 - \frac{32}{5}\right) = \frac{16}{5}\pi$$

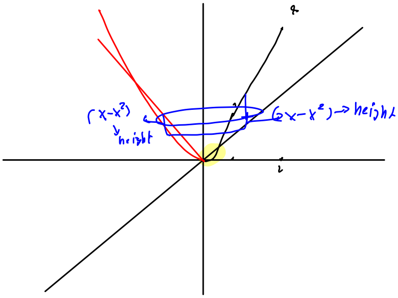


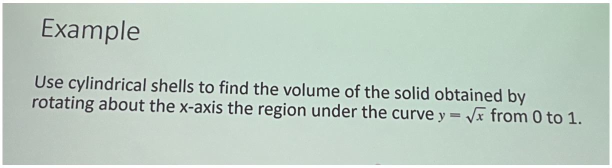
Example

Find the volume of the solid obtained by rotating about the y-axis the region between y = x and y = x².

$$V = \int_0^1 (2\pi x)(x - x^2)\,dx = 2\pi \int_0^1 (x^2 - x^3)\,dx$$

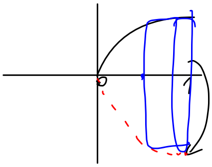
$$= 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{\pi}{6}$$



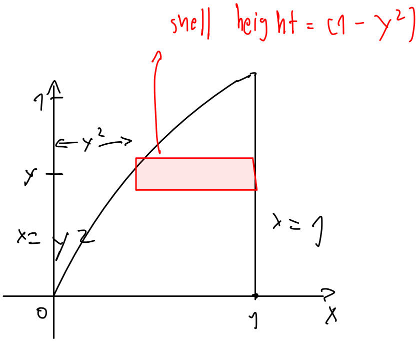


Example

Use cylindrical shells to find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.



$x = y^2$



$$= v = \int_0^1 2\pi y (1 - y^2)\,dy$$

$$= 2\pi \int_0^1 y - y^3$$

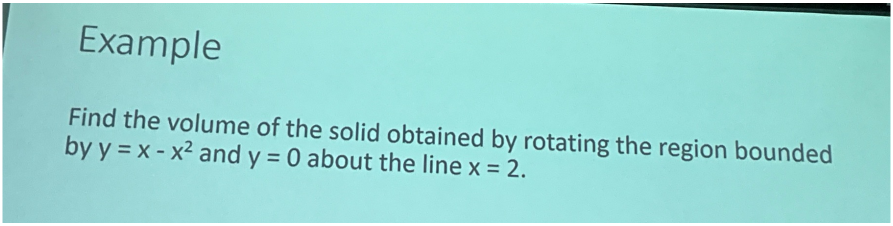
$$= 2\pi \left[\frac{y^2}{2} - \frac{y^4}{4} \right]_0^1 = \frac{\pi}{2}$$

Summary for Cylindrical shell

Radius use height instead

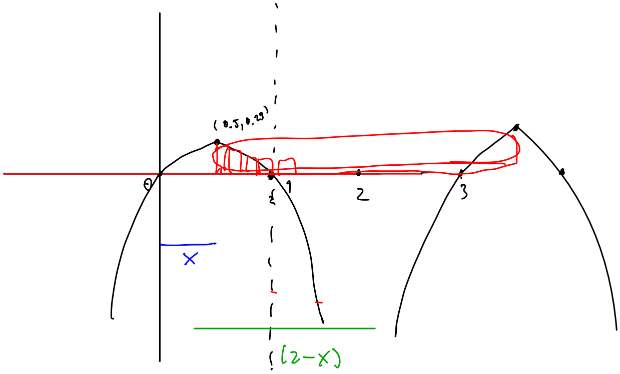
$$A(x) = \int_A^b 2\pi x \cdot f(x)\,dx$$

#



Example

Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.

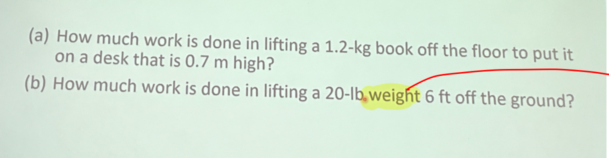


$r = 2 - x$

$A(x) = 2\pi(2-x)(x-x^2)$

Work

→ $W = F \times d$, $F = ma = m \frac{d^2 s}{dt^2}$



(a) How much work is done in lifting a 1.2-kg book off the floor to put it on a desk that is 0.7 m high?

(b) How much work is done in lifting a 20-lb weight 6 ft off the ground?

→ Note: weight is a force.

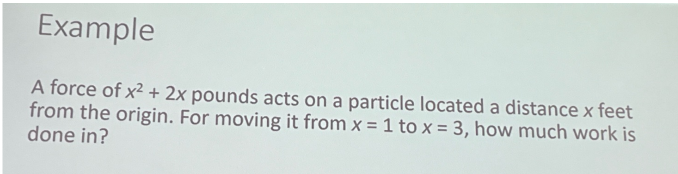
a. $F = ma = (1.2)(9.8) = 11.76$ N

$W = Fd = (11.76)(0.7\text{m}) = 8.2$ J

b. $W = Fd = (20\text{lb})(6\text{ft}) = 120$ ft-lb.

What if the force is not constant.

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \int_a^b f(x)\,dx$$



Example

A force of $x^2 + 2x$ pounds acts on a particle located a distance x feet from the origin. For moving it from $x = 1$ to $x = 3$, how much work is done in?

$(x^2+2x)\frac{dx}{ds}$

$$W = \int_1^3 x^2 + 2x$$

$$= \frac{x^3}{3} + x^2 \Big|_1^3$$

$$= \left[\frac{27}{3} + 9\right] - \left[\frac{1}{3} + 1\right]$$

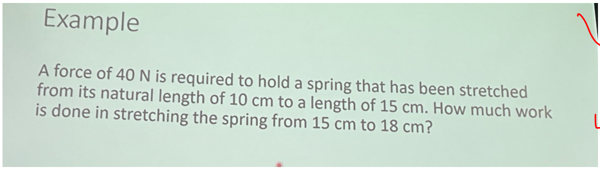
need to know conversion.

$$= 18 - \frac{4}{3} = \frac{50}{3} \text{ ft-lb}$$

ft

Hooke Law

$f(x) = kx$



Example

A force of 40 N is required to hold a spring that has been stretched from its natural length of 10 cm to a length of 15 cm. How much work is done in stretching the spring from 15 cm to 18 cm?

Note: convert to M first. (meter)

$5 cm = 0.05 m$ $f(0.05) = 40$ $f(x) = kx$ $\cdot\ 40 = k(0.05$

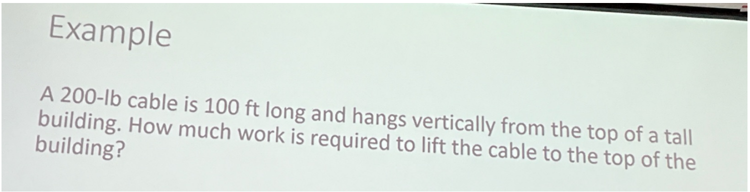
$k = 800\ \cdot$

$0.05k = 40$ $k = \frac{40}{0.05} = 800$ $f(x) = 800x$

$$W = \int_{0.05 \to (0.15 - 0.1)}^{0.08 \to (0.18 - 0.1)} 800x\,dx = 1.56 J$$

(0.18 − 0.1) → natural length

(0.15 − 0.1) → natural length

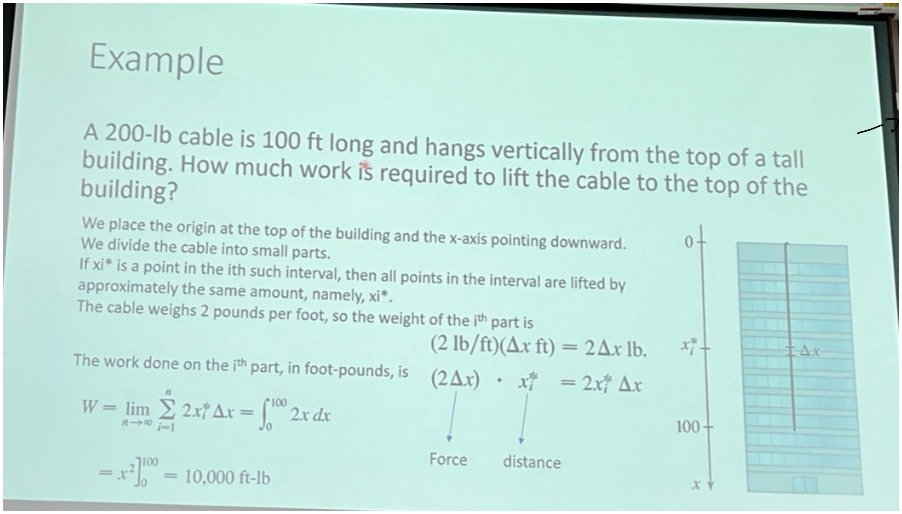


Example

A 200-lb cable is 100 ft long and hangs vertically from the top of a tall building. How much work is required to lift the cable to the top of the building?

!! Force ถ้าเกิดเป็น vs unit ต้องเป็น -lb/ft

$= \quad F = ma = \quad F = 200$



Example

A 200-lb cable is 100 ft long and hangs vertically from the top of a tall building. How much work is required to lift the cable to the top of the building?

We place the origin at the top of the building and the x-axis pointing downward.
We divide the cable into small parts.
If xi* is a point in the ith such interval, then all points in the interval are lifted by approximately the same amount, namely, xi*.
The cable weighs 2 pounds per foot, so the weight of the ith part is

$$(2\ \text{lb/ft})(\Delta x\ \text{ft}) = 2\Delta x\ \text{lb.}$$

The work done on the ith part, in foot-pounds, is

$$\underbrace{(2\Delta x)}_{\text{Force}} \cdot \underbrace{x_i^*}_{\text{distance}} = 2x_i^*\,\Delta x$$

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} 2x_i^*\,\Delta x = \int_0^{100} 2x\,dx$$

$$= x^2\Big]_0^{100} = 10{,}000\ \text{ft-lb}$$

$\to F = \frac{200\ lb}{100\ ft}$

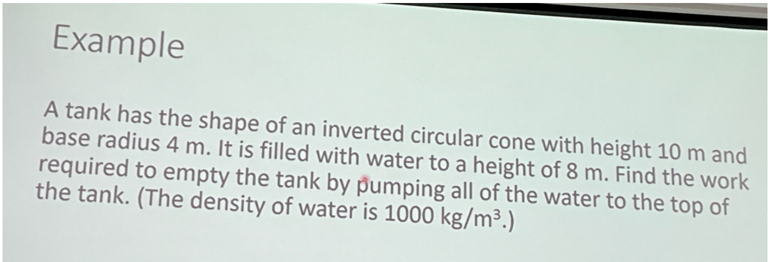
$= 2\ lb/ft$

$= 2 lb/ft\ x$

$= f = kx$

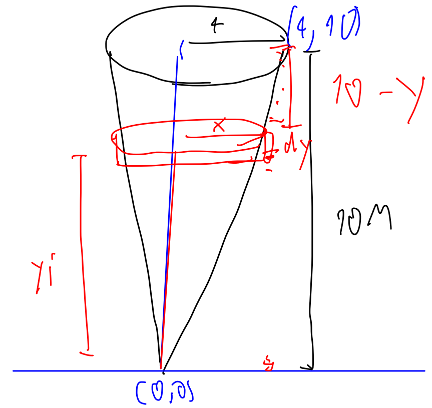
$f = 2x$

$f(x) = 2x$



Example

A tank has the shape of an inverted circular cone with height 10 m and base radius 4 m. It is filled with water to a height of 8 m. Find the work required to empty the tank by pumping all of the water to the top of the tank. (The density of water is 1000 kg/m³.)



Work = Distance × Density × Volume

$$\text{Volume} = \pi r^2 \cdot \text{thickness}$$

$$= \pi (x)^2 \cdot dy$$

$$\text{Density} \overset{(\text{siunit})}{=} 1000 \times 9.8$$

$$\text{Distance} = (10 - y)$$

Find x

$$\left(\frac{10-0}{4-0}\right) = \frac{5}{2}$$

$$y - y_1 = m(x - x_1)$$

$$y - 0 = \frac{5}{2}(x - 0)$$

$$y = \frac{5}{2}x \qquad x = \frac{2}{5}y$$

$$= \int_0^8 (10-y)(1000 \times 9.8) \cdot \left(\pi \left(\frac{2}{5}y\right)^2\right) \cdot dy$$

$$= 9800\pi \int_0^8 (10 - y) \cdot \frac{4}{25}y^2 \, dy$$

$$= 9800\pi \int_0^8 \frac{40}{25}y^2 - \frac{4}{25}y^3 \, dy$$

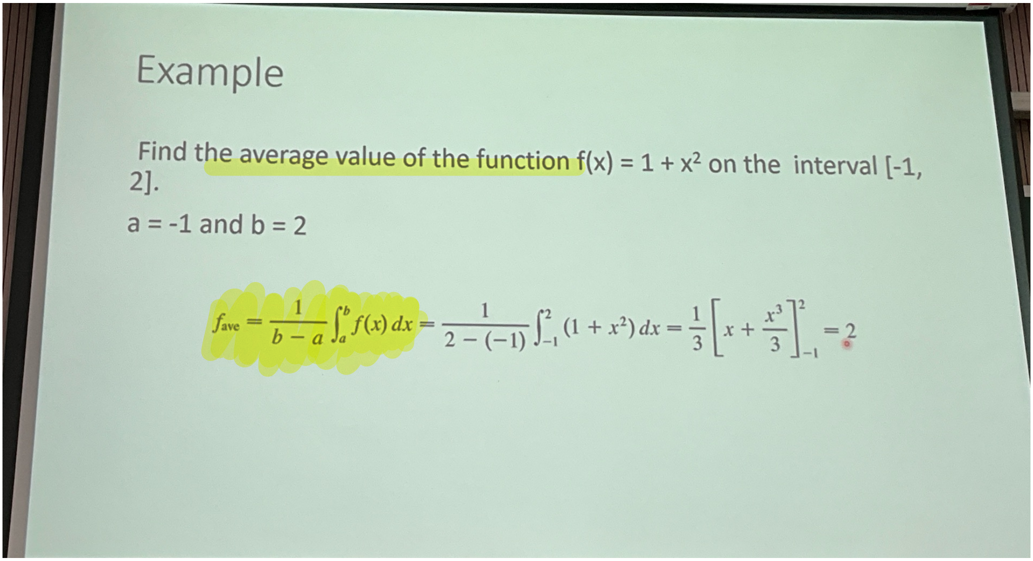
$$= 3362827.747 \text{ J}$$

\#

Average Value of a Function

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\lim_{n\to\infty} \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\,\Delta x = \frac{1}{b-a}\int_a^b f(x)\,dx$$



Example

Find the average value of the function f(x) = 1 + x² on the interval [-1, 2].

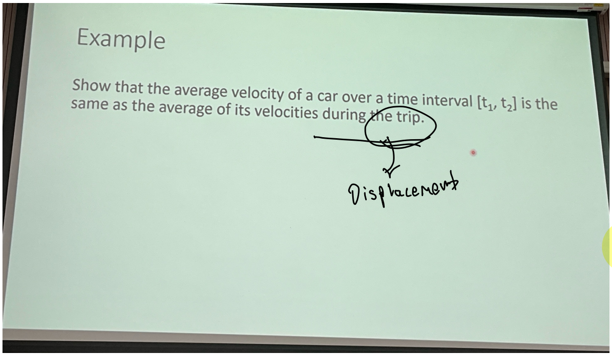
a = -1 and b = 2

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{2-(-1)}\int_{-1}^{2}(1+x^2)\,dx = \frac{1}{3}\left[x+\frac{x^3}{3}\right]_{-1}^{2} = 2$$

Theorem: The mean Value Theorem

If f is continuous on [a, b], then there exists a number c in [a, b] such that

$$f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx \longrightarrow \int_a^b f(x)\,dx = f(c)(b-a)$$



Example

Show that the average velocity of a car over a time interval $[t_1, t_2]$ is the same as the average of its velocities during the trip.

Displacement

If s(t) is the displacement of the car at time t, then, the

$$\text{average velocity} = \frac{\Delta s}{\Delta t} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

↓

$\frac{\text{Displacement}}{\text{time}}$

$$= \frac{1}{t_2 - t_1}\int_{t}^{t_2} v(t)\,dt = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} s'(t)\,dt = \frac{1}{t_2 - t_1}\left[s(t_2) - s(t_1)\right]$$

$$= \frac{s(t_2) - s(t_1)}{t_2 - t_1} \nearrow \text{AVG Velocity}$$

$\frac{dy}{dt}$ → Displacement = Velocity #

⇣

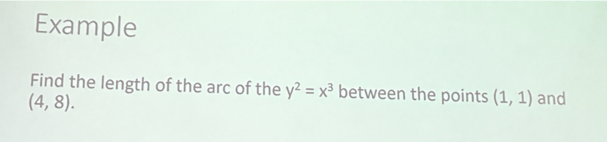
s'(t)

Arclength

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} |P_{i-1} P_i|$$

If f' is continuous on [a, b], then the length of the curve $y = f(x)$, $a<x<b$ is

$$= L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx \quad \text{or} \quad L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$



Example

Find the length of the arc of the $y^2 = x^3$ between the points (1, 1) and (4, 8).

$y = x^{\frac{3}{2}}$ from (1, 4)

$$= \int_1^4 \sqrt{1 + \left(\frac{3}{2}x^{\frac{1}{2}}\right)^2}\,dx = \int_1^4 \sqrt{1 + \frac{9}{4}x}$$

↓

$v = 1 + \frac{9}{4}x$

when $x = 1$, $v = \frac{13}{4}$

$x = 4$, $v = 10$

↓

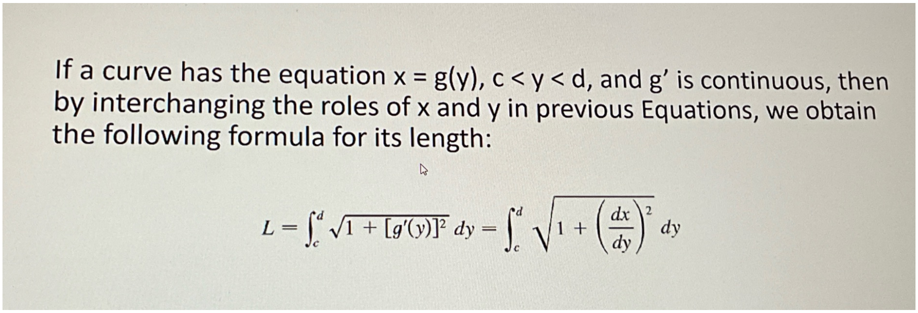
$dv = \frac{9}{4}\,dx$ $dx = \frac{4}{9}\,dv$

$= \int_1^4 \sqrt{v} \cdot \frac{4}{9}\,dv = \frac{4}{9}\int_{\frac{13}{4}}^{10} \sqrt{v}\,dv$

↓

$= \frac{4}{9} \cdot \left[\frac{2}{3} v^{\frac{3}{2}}\right]_{\frac{13}{4}}^{10}$

$= \frac{1}{27}\left(80\sqrt{10} - 13\sqrt{13}\right)$



If a curve has the equation x = g(y), c < y < d, and g’ is continuous, then by interchanging the roles of x and y in previous Equations, we obtain the following formula for its length:

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\,dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$$

Ex. Find the length of the arc of parabola $y^2 = x$ from $(0,0)$ to $(1,1)$.

$L = \int_0^1 \sqrt{1 + (2y)^2}\,dy = \int_0^1 \sqrt{\frac{1+4y^2}{4}}\,dy$

$\frac{dx}{dy}$

$= \int_0^1 \sqrt{\frac{1}{4} + y^2}$

$= y = \frac{1}{2}\tan\theta$ $dy = \frac{1}{2}\sec^2\theta\,d\theta$

↓

$$= \sqrt{1 + 4y^2}$$

$$= \sqrt{1 + 4\left(\tfrac{1}{2}\tan\theta\right)^2} = \sqrt{1 + \tan^2\theta} = \sqrt{\sec^2\theta} = \sec\theta$$

$$= \int_0^1 \sec\theta \cdot \frac{1}{2}\sec^2\theta \, d\theta$$

when $y = 0$ $\tan\theta = 0$

so $\theta = 0$

when $y = 1$, $\tan\theta = 2$

so $= \theta = \tan^{-1} 2 = \alpha$

$$= \frac{1}{2}\int_0^1 \sec^3\theta \, d\theta$$

$$= \frac{1}{2}\left[\int_0^1 \sec^2\theta \cdot \sec\theta \, d\theta \right.$$

$u = \sec\theta$ $\quad du = \sec\theta\tan\theta \, d\theta$

$dv = \sec^2\theta \, d\theta$ $\quad v = \tan\theta$

$$= \tan\theta\sec\theta - \int \tan\theta \cdot \sec\theta\tan\theta \, d\theta$$

$$= \tan\theta \sec\theta - \int \tan^2\theta \sec\theta \, d\theta$$

$$\downarrow$$

$$= \int \sec^3 x - \sec x \, dx$$

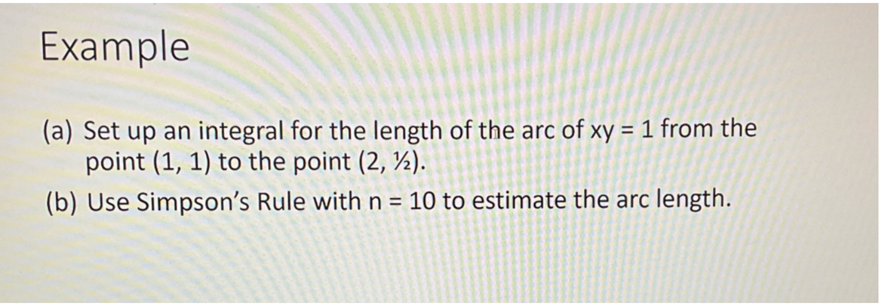
$$I = \tan\theta \sec\theta - I + \int \sec x \, dx$$

$$2I = \tan\theta\sec\theta + \ln|\sec x + \tan x|$$

$$\frac{1}{2} = \left[\frac{1}{2}\tan\theta\sec\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|\right]_0^\alpha$$

$$L = \frac{\sqrt{5}}{2} + \frac{\ln(\sqrt{5} + 2)}{4}$$

#



Example

(a) Set up an integral for the length of the arc of xy = 1 from the point (1, 1) to the point (2, ½).

(b) Use Simpson's Rule with n = 10 to estimate the arc length.

a. $y = \frac{1}{x}$ point (1, 1) to $(2, \frac{1}{2})$

$\frac{dy}{dx}$ of $\frac{1}{x} = -x^{-2}$

$$\int_1^2 \sqrt{1+(-x^{-2})^2}\;dx$$

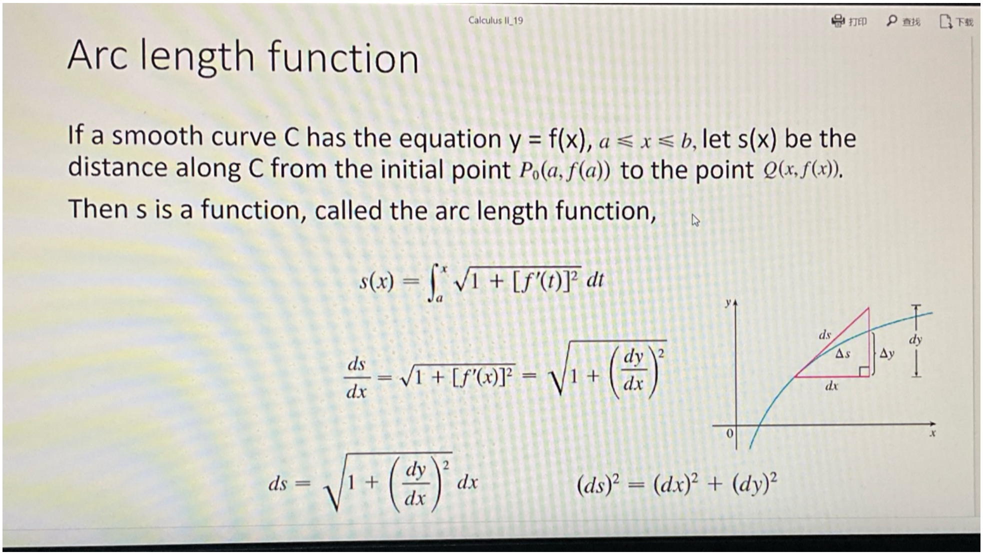
$$= \int_1^2 \sqrt{1+\frac{1}{x^4}}\,dx = \int_1^2 \frac{\sqrt{x^4+1}}{x^2}\,dx$$

b). $a = 1$, $b = 2$ $n = 10$ $\Delta x = \frac{2-1}{10} = 0.1$, $f(x) = \sqrt{1+\frac{1}{x^4}}$

$$= \frac{\Delta x}{3}\left[f(1) + 4(f(1.1)) + 2f(1.2) + 4f(1.3) + \dots + 2f(1.8) + 4f(1.9) + f(2)\right]$$

≈ 1.1321

Arclength Function



Arc length function

If a smooth curve C has the equation y = f(x), $a \leqslant x \leqslant b$, let s(x) be the distance along C from the initial point $P_0(a, f(a))$ to the point $Q(x, f(x))$.

Then s is a function, called the arc length function,

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt$$

$$\frac{ds}{dx} = \sqrt{1 + [f'(x)]^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad (ds)^2 = (dx)^2 + (dy)^2$$

Find arc length function for $y = x^2 - \frac{1}{8}\ln x$ taking $P_0(1, 1)$ as the starting point.

$$f(x) = x^2 - \frac{1}{8}\ln x \qquad f'(x) = 2x - \frac{1}{8x}$$

$$1 + f'(x)^2 = 1 + \left(2x - \frac{1}{8x}\right)^2 = 1 + 4x^2 - \frac{1}{2} + \frac{1}{64x^2}$$

$$= 4x^2 + \frac{1}{2} + \frac{1}{64x^2} = \left(2x + \frac{1}{8x}\right)^2$$

$$= \sqrt{\left(2x + \frac{1}{8x}\right)^2}$$

$$= 2x + \frac{1}{8x}$$

✓

$$S(x) = \int_1^x \sqrt{1+[f'(t)]^2}\, dt$$

$$= \int_1^x \left(2t + \frac{1}{8t}\right) dt$$

$$= t^2 + \frac{1}{8}\ln t \Big]_1^x = x^2 + \frac{1}{8}\ln x - 1$$

//

Area of a surface of Revolution

Area for surface revolution for cone $= \pi r l$

$\downarrow$

cylinder $= 2\pi r h$

Surface area can be approximated as

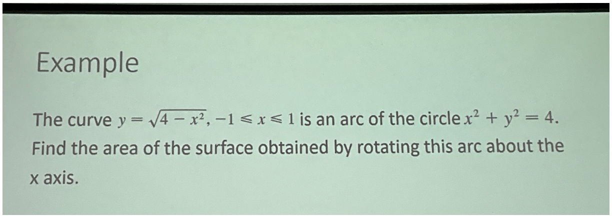
$$\sum_{i=1}^{n} 2\pi f(x_i^*) \sqrt{1+[f'(x_i^*)]^2}\, \Delta x$$

$\downarrow$

$$S = \int_a^b 2\pi f(x) \sqrt{1+[f'(x)]^2}\, dx$$

$\downarrow$

$$S = \int 2\pi y \, ds \quad \text{or} \quad \int 2\pi x \, ds$$



Example

The curve $y = \sqrt{4 - x^2}$, $-1 \leqslant x \leqslant 1$ is an arc of the circle $x^2 + y^2 = 4$. Find the area of the surface obtained by rotating this arc about the x axis.

$$y = \sqrt{4 - x^2} \qquad \frac{dy}{dx} = \frac{1}{\cancel{2}}(4-x^2)^{-\frac{1}{2}} \cdot -\cancel{2}x$$

$$= \frac{-x}{\sqrt{4-x^2}}$$

$$2\pi \int_{-1}^{1} y\sqrt{1+\left(\frac{-x}{\sqrt{4-x^2}}\right)^2}\, dx$$

$$= 2\pi \int_{-1}^{1} y\sqrt{1+\frac{x}{4-x^2}}\, dx$$

$$= 2\pi \int_{-1}^{1} y\sqrt{\frac{4-x^2+x}{4-x^2}}\, dx$$

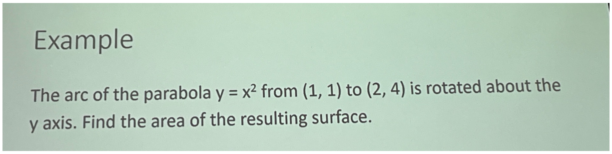
$$= 2\pi \int_{-1}^{1} y\sqrt{\frac{4}{4-x^2}}\, dx$$

$$= 2\pi \int_{-1}^{1} \cancel{\sqrt{4-x^2}}\, \frac{2}{\cancel{\sqrt{4-x^2}}}\, dx$$

$$= 4\pi \int_{-1}^{1} 1 \, dx$$

$$= 4\pi\,(1-(-1)) = 4\pi(2) = 8\pi$$

#



Example

The arc of the parabola y = x² from (1, 1) to (2, 4) is rotated about the y axis. Find the area of the resulting surface.

$$= 2\pi \int_1^2 x \sqrt{1 + (2x)^2}\; dx$$

$$= 2\pi \int_1^2 x \sqrt{1 + 4x^2}\; dx$$

$$u = 1 + 4x^2 \qquad du = 8x\, dx$$

$$dx = \frac{du}{8x}$$

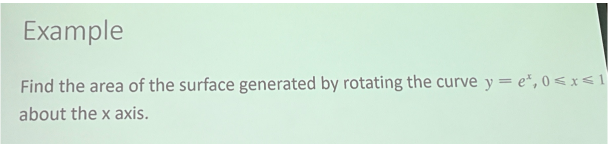
$$= 2\pi \int_1^2 \cancel{x} \sqrt{u} \cdot \frac{du}{8\cancel{x}}$$

$$= \frac{1}{4}\pi \int_1^2 \sqrt{u}\; du$$

$$= \frac{1}{4}\pi \left(\frac{2(1+4x^2)^{\frac{3}{2}}}{3} \Bigg|_1^2 \right)$$

$$= \frac{\pi}{16}\left(17\sqrt{17} - 5\sqrt{5}\right)$$

#



Example

Find the area of the surface generated by rotating the curve $y = e^x$, $0 \leqslant x \leqslant 1$ about the x axis.

$$2\pi \int_0^1 e^x \cdot \sqrt{1+(e^x)^2}\ dx$$

$$= 2\pi \int_0^1 e^x \sqrt{1+e^{2x}}\ dx$$

$$v = e^x \qquad dv = e^x\, dx$$

$$dx = \frac{dv}{e^x}$$

$$= 2\pi \int_1^e \cancel{e^x} \sqrt{1+v^2} \cdot \frac{dv}{\cancel{e^x}}$$

$$= 2\pi \int_1^e \sqrt{1+x^2}\ dx$$

$$x = \tan\theta \qquad dx = \sec^2\theta\ d\theta$$

$$= 2\pi \int_1^e \sec\theta \cdot \sec^2\theta\ d\theta$$

$v = \tan\theta$ and $\alpha = \tan^{-1} e$

so if $v = 1$ $\theta = \tan^{-1}(1) = \frac{\pi}{4}$

$$= 2\pi \int_{\frac{\pi}{4}}^{\alpha} \sec^3\theta\ d\theta$$

$$v = \sec\theta \qquad du = \sec\theta\tan\theta\ d\theta$$

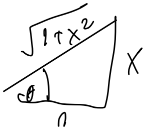
$$dv = \sec^2\theta\ d\theta \qquad v = \tan\theta$$

$$= \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\ d\theta$$

$\int \sec^3\theta - \sec\theta$

$I = \sec\theta\tan\theta - I + \ln|\sec\theta + \tan\theta|$

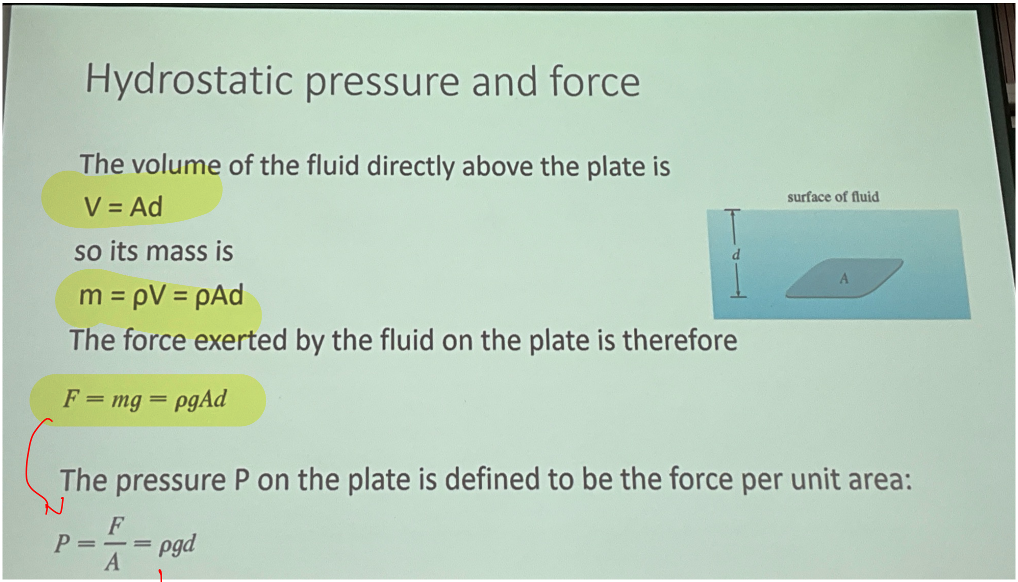
$2\pi \cdot \frac{1}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right]\Big]_{\frac{\pi}{4}}^{d}$



$= \pi\left[\sqrt{1+x^2} \cdot x + \ln|\sqrt{1+x^2} + x|\right]_{\frac{\pi}{4}}^{d}$

$S = \pi\left[e\sqrt{1+e^2} + \ln(e + \sqrt{1+e^2}) - \sqrt{2} - \ln(\sqrt{2}+1)\right]$

Hydrostatic pressure and Force



Hydrostatic pressure and force

The volume of the fluid directly above the plate is

V = Ad

so its mass is

m = ρV = ρAd

The force exerted by the fluid on the plate is therefore

$F = mg = \rho g A d$

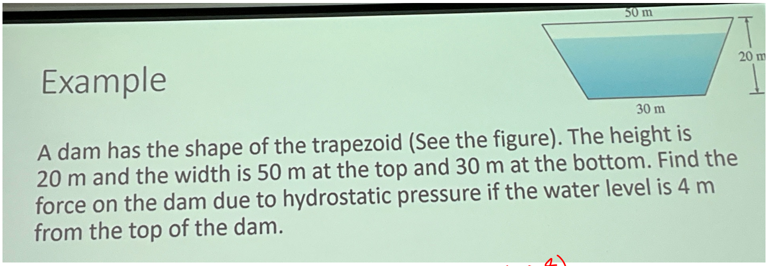
The pressure P on the plate is defined to be the force per unit area:

$P = \frac{F}{A} = \rho g d$

P=(1000)(9.8)(distance)

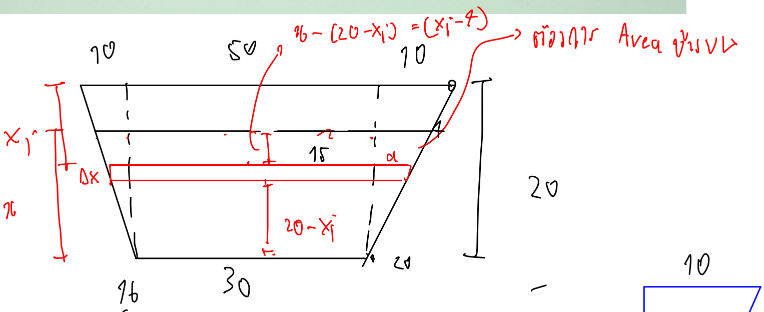
normally y

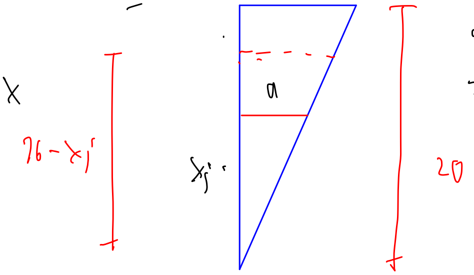
When find a force, use F = P×A and plug in P.



Example

A dam has the shape of the trapezoid (See the figure). The height is 20 m and the width is 50 m at the top and 30 m at the bottom. Find the force on the dam due to hydrostatic pressure if the water level is 4 m from the top of the dam.





$$= 1000(9.8)\int_0^{16} (16 - x_i)(30 + x_i)\,\Delta x$$

$$= 9800\int_0^{16} 480 + 16x - 30x - x^2 \; dx$$

$$= 9800\int_0^{16} 480 - 14x - x^2 \; dx$$

$$= 9800\left[480x - 7x^2 - \frac{x^3}{3}\right]_0^{16}$$

$\approx$ 44,322,133.33 N #

$x(46 - x_i)$

$\frac{a}{x_i} = \frac{10}{20}$

$a = \frac{1}{2}x$

$a = x$

$30 + x_i'$

$\frac{a}{16 - x_i} = \frac{10}{20}$

$a = \frac{\cancel{10}(16 - x_i)}{\cancel{20}_2}$

$x = 30 + \cancel{2}\left(\frac{16 - x_i}{\cancel{2}}\right)$

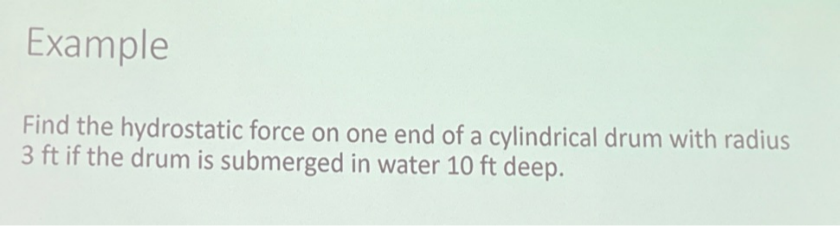
$= 30 + 16 - x_i$

$= 46 - x_i$

Useful Formula

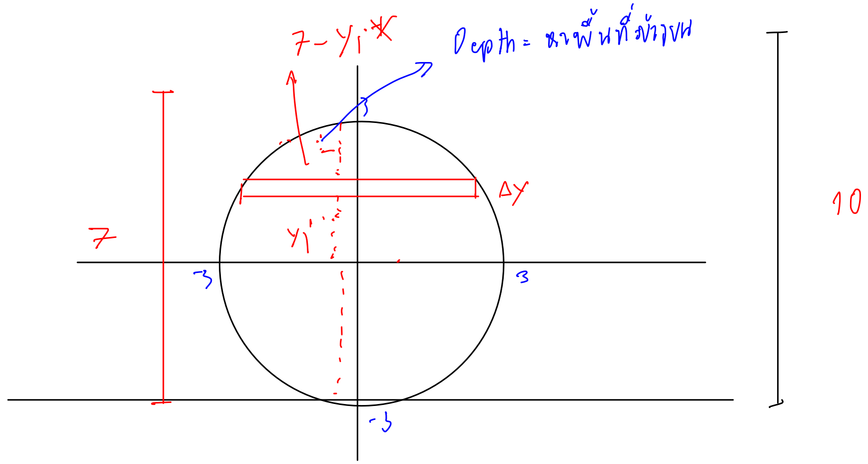
$$F = \underbrace{w}_{\text{Weight Density}} \int_{c}^{d} \underbrace{h(y)}_{\text{depth}} \; \underbrace{L(y)}_{\text{length}} \; \underbrace{dy}_{\text{width}} \longrightarrow \text{For hydrostatic force}$$

เริ่มดูจากข้างบน จนมาถึงพื้นที่ตัดที่เราเลือก



Example

Find the hydrostatic force on one end of a cylindrical drum with radius 3 ft if the drum is submerged in water 10 ft deep.



$x^2+y^2=9$

$x^2=9-y^2$

$x=\sqrt{9-(y_i)^2}$

Area $= 2\sqrt{9-(y_i)^2}\cdot \Delta y$

$P = 62.4\ lb/ft \cdot (7-y_i)$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} 62.4(7-y_i)\,2\sqrt{9-(y_i)^2}\,\Delta y$$

$$125\int_{-3}^{3}(7-y)\sqrt{9-y^2}\,dy \to 125\cdot 7\int_{-3}^{3}\sqrt{9-y^2}\,dy - 125\int_{-3}^{3} y\sqrt{9-y^2}\,dy$$

odd function so integral = 0

$$F = 875\int_{-3}^{3}\sqrt{9-y^2}\,dy = 875\cdot\frac{1}{2}\pi(3)^2$$

use $y=3\sin\theta$

$$= \frac{7875\pi}{2} \approx 12,370\ lb$$

$$F = 875 \int_{-3}^{3} \sqrt{9-y^2}\, dy \qquad dy = 3\cos\theta\, d\theta$$

$$= 875 \int_{-3}^{3} \sqrt{9-y^2}\, dy$$

$$y = 3\sin\theta \qquad dy = 3\cos\theta\, d\theta$$

$$= 875(2) \int_{0}^{3} \sqrt{9-(3\sin\theta)^2} \cdot 3\cos\theta\, d\theta$$

$$= 1750 \int_{0}^{3} \sqrt{9(1-\sin^2\theta)}\ 3\cos\theta\, d\theta$$

$$= 1750 \int_{0}^{3} 3\cos\theta \cdot 3\cos\theta\, d\theta$$

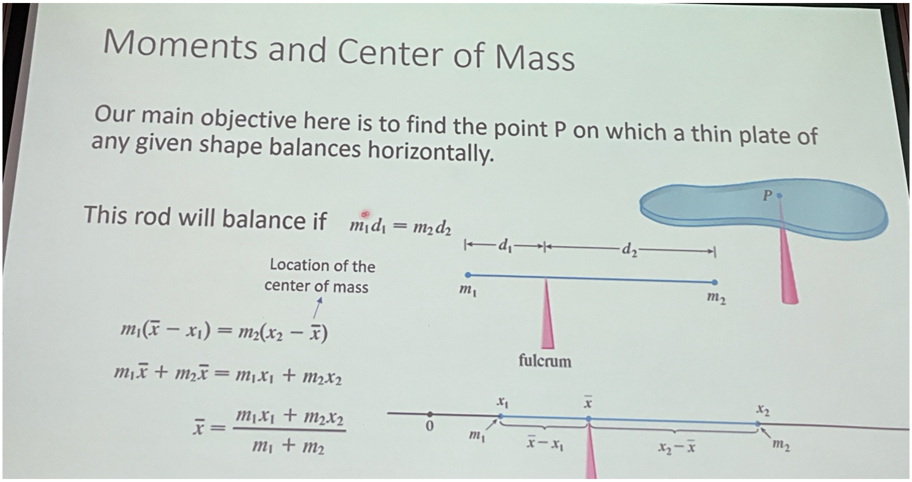
$$= 15750 \int_{0}^{3} \cos^2\theta\, d\theta$$

$$= 15750 \int_{0}^{3} \frac{1+\cos 2\theta}{2} \rightarrow \text{cos identity}$$

$$= 15750 \int_{0}^{\frac{\pi}{2}} \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) \rightarrow = \frac{7875}{2}\pi$$

#

Moments and Center of Mass



Moments and Center of Mass

Our main objective here is to find the point P on which a thin plate of any given shape balances horizontally.

This rod will balance if $m_1 d_1 = m_2 d_2$

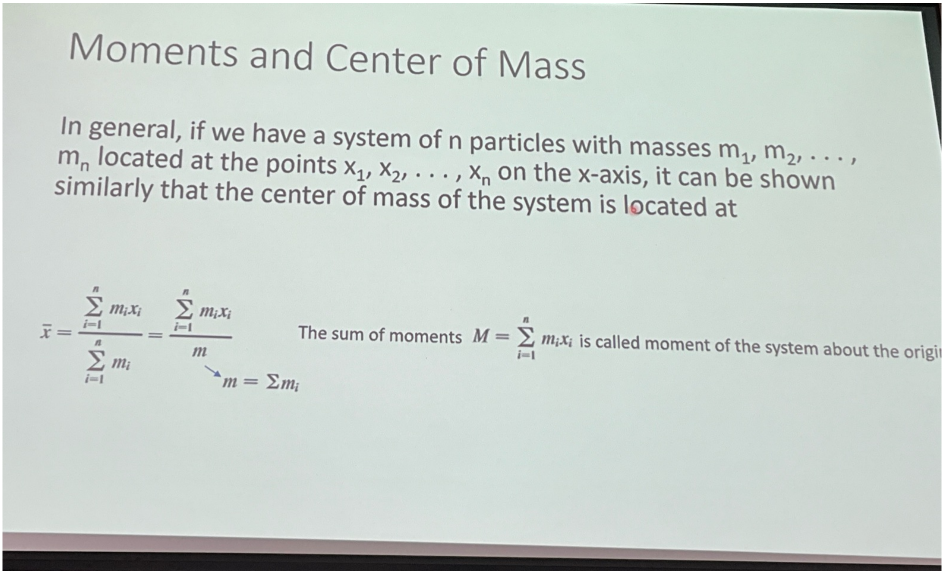
Location of the center of mass

$$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$$

$$m_1\bar{x} + m_2\bar{x} = m_1x_1 + m_2x_2$$

$$\bar{x} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

$m_1 x_1$ and $m_2 x_2$ = Moments of the masses



Moments and Center of Mass

In general, if we have a system of n particles with masses $m_1, m_2, \ldots,$ m_n located at the points $x_1, x_2, \ldots, x_n$ on the x-axis, it can be shown similarly that the center of mass of the system is located at

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} m_i x_i}{m}$$

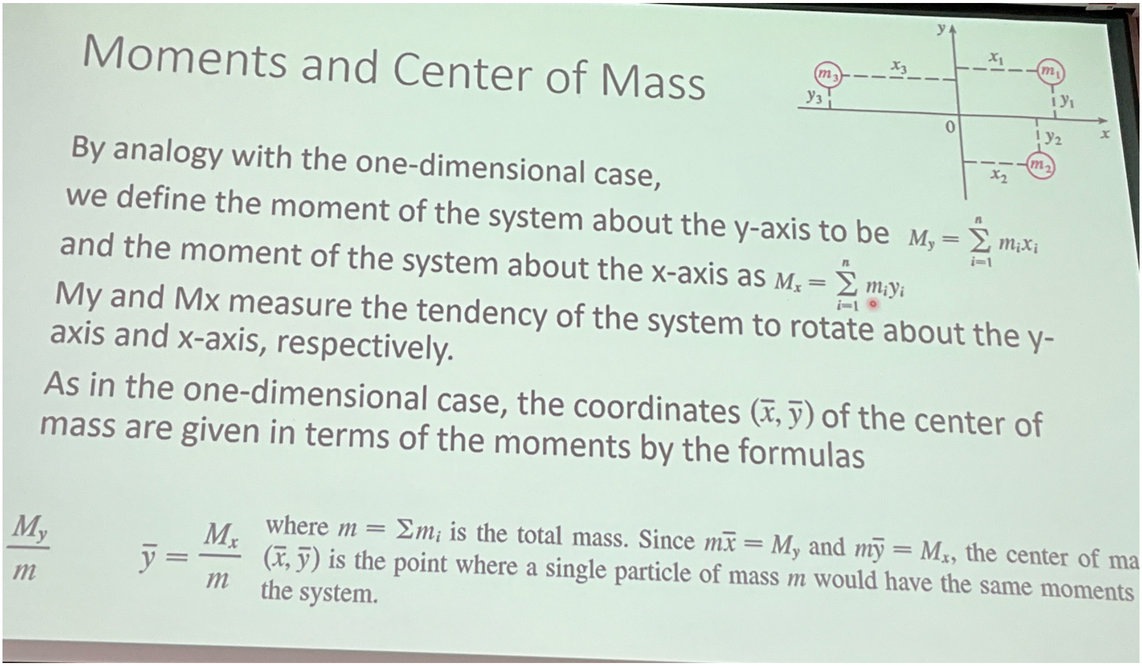
$m = \Sigma m_i$

The sum of moments $M = \sum_{i=1}^{n} m_i x_i$ is called moment of the system about the origi

Sum of moments = $M = \sum_{i=1}^{n} m_i X_i$ } Moment of system about the origin

↓

$$m\bar{x} = M$$



Moments and Center of Mass

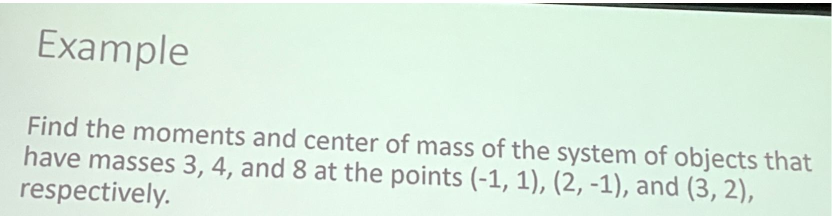
By analogy with the one-dimensional case, we define the moment of the system about the y-axis to be $M_y = \sum_{i=1}^{n} m_i x_i$ and the moment of the system about the x-axis as $M_x = \sum_{i=1}^{n} m_i y_i$

My and Mx measure the tendency of the system to rotate about the y-axis and x-axis, respectively.

As in the one-dimensional case, the coordinates $(\bar{x}, \bar{y})$ of the center of mass are given in terms of the moments by the formulas

$$\frac{M_y}{m} \qquad \bar{y} = \frac{M_x}{m}$$

where $m = \Sigma m_i$ is the total mass. Since $m\bar{x} = M_y$ and $m\bar{y} = M_x$, the center of ma $(\bar{x}, \bar{y})$ is the point where a single particle of mass m would have the same moments the system.



Example

Find the moments and center of mass of the system of objects that have masses 3, 4, and 8 at the points (-1, 1), (2, -1), and (3, 2), respectively.

moment

$M_y = 3(-1) + 4(2) + 8(3) = 29$

$M_x = 3(1) + 4(-1) + 8(2) = 15$

$m = 3+4+8 = 15$

Moments and center of mass

$\bar{x} = \frac{M_y}{m} = \frac{29}{15}$ $\quad \bar{y} = \frac{M_x}{m} = \frac{15}{15} = 1$

!! Center of mass eq.

$\bar{x} = \frac{M_y}{m}$, $\bar{y} = \frac{M_x}{m}$

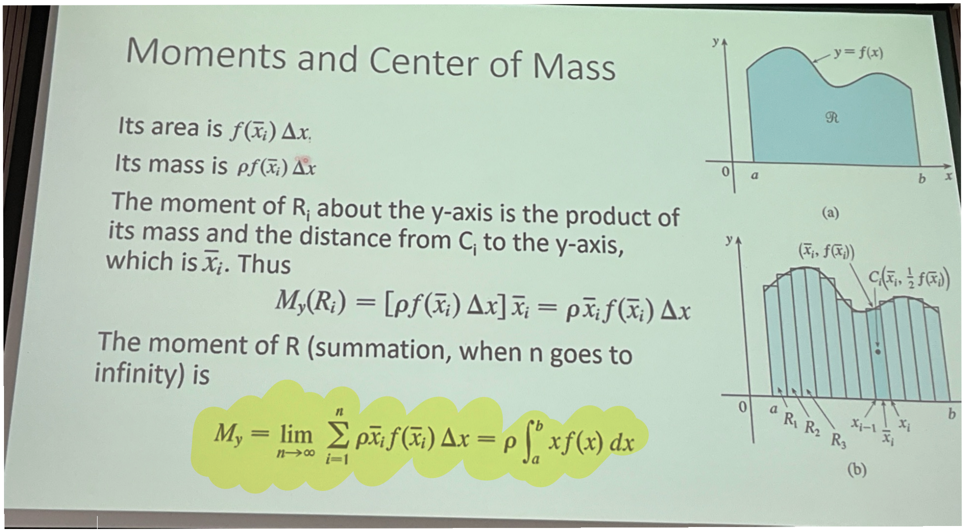
$M_y = \sum_{i=1}^{n} m_i x_i$

$M_x = \sum_{i=1}^{n} m_i y_i$

Mass = 3+4+8 = 15

Center of mass = $\bar{x} = \frac{M_y}{m} = \frac{29}{15}$, $\bar{y} = \frac{M_x}{m} = \frac{15}{15} = 1$ = Ans = $(\frac{29}{15}, 1)$

Some time not that easy



Moments and Center of Mass

Its area is $f(\bar{x}_i)\,\Delta x$.

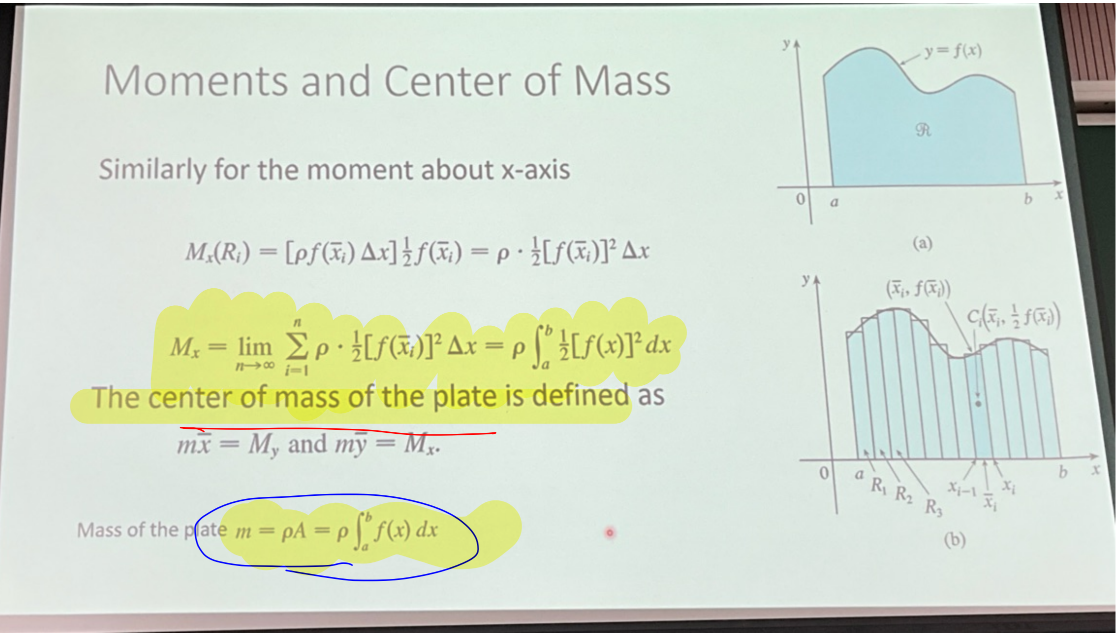
Its mass is $\rho f(\bar{x}_i)\,\Delta x$

The moment of R_i about the y-axis is the product of its mass and the distance from C_i to the y-axis, which is $\bar{x}_i$. Thus

$$M_y(R_i) = [\rho f(\bar{x}_i)\,\Delta x]\,\bar{x}_i = \rho\bar{x}_i f(\bar{x}_i)\,\Delta x$$

The moment of R (summation, when n goes to infinity) is

$$M_y = \lim_{n\to\infty}\sum_{i=1}^{n}\rho\bar{x}_i f(\bar{x}_i)\,\Delta x = \rho\int_a^b x f(x)\,dx$$



Moments and Center of Mass

Similarly for the moment about x-axis

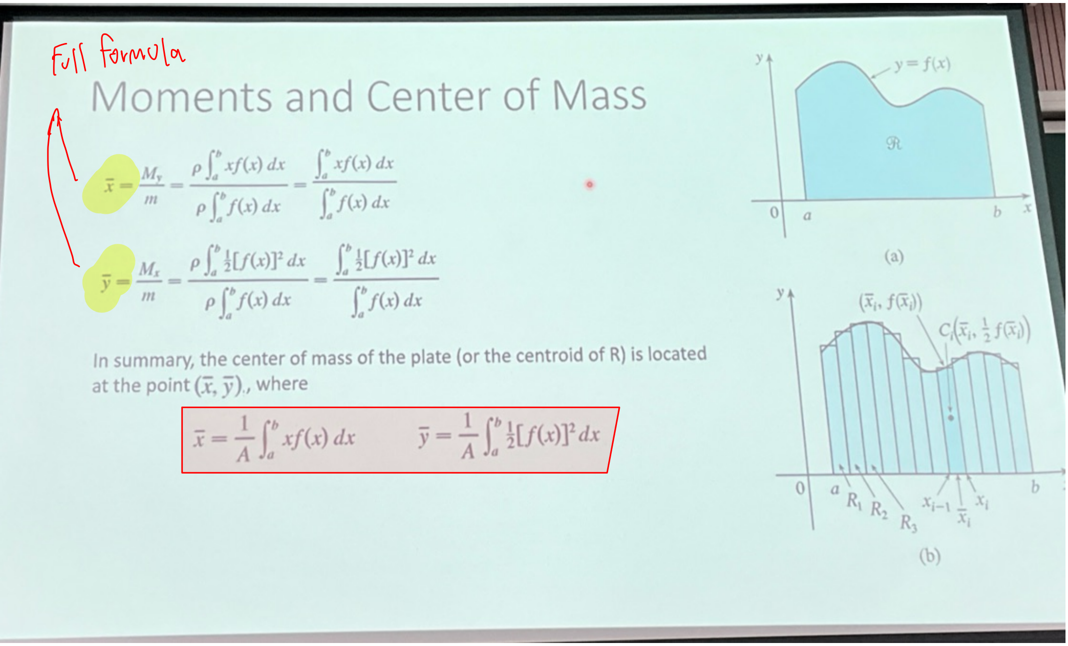
$$M_x(R_i) = [\rho f(\bar{x}_i)\,\Delta x]\,\tfrac{1}{2}f(\bar{x}_i) = \rho\cdot\tfrac{1}{2}[f(\bar{x}_i)]^2\,\Delta x$$

$$M_x = \lim_{n\to\infty}\sum_{i=1}^{n}\rho\cdot\tfrac{1}{2}[f(\bar{x}_i)]^2\,\Delta x = \rho\int_a^b \tfrac{1}{2}[f(x)]^2\,dx$$

The center of mass of the plate is defined as

$m\bar{x} = M_y$ and $m\bar{y} = M_x$.

Mass of the plate $m = \rho A = \rho\int_a^b f(x)\,dx$



Full formula

Moments and Center of Mass

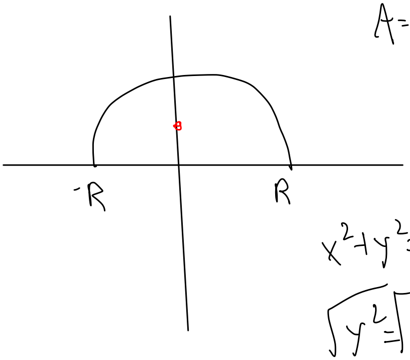
$$\bar{x} = \frac{M_y}{m} = \frac{\rho \int_a^b x f(x)\, dx}{\rho \int_a^b f(x)\, dx} = \frac{\int_a^b x f(x)\, dx}{\int_a^b f(x)\, dx}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\rho \int_a^b \frac{1}{2}[f(x)]^2\, dx}{\rho \int_a^b f(x)\, dx} = \frac{\int_a^b \frac{1}{2}[f(x)]^2\, dx}{\int_a^b f(x)\, dx}$$

In summary, the center of mass of the plate (or the centroid of R) is located at the point $(\bar{x}, \bar{y})$., where

$$\boxed{\bar{x} = \frac{1}{A}\int_a^b x f(x)\, dx \qquad \bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2\, dx}$$

Find the center of mass of a semicircular plate of radius. R



$A = \frac{1}{2}\pi r^2 \quad dx$

$A = \frac{1}{2}\pi(\sqrt{r^2 - x^2})^2$

$x^2 + y^2 = R^2$

$\sqrt{y^2} = \sqrt{R^2 - x^2}$

$y = \sqrt{r^2 - x^2}$

Center should lie on y-axis so use $\bar{y}$ formula

$$= \frac{1}{\frac{1}{2}\pi r^2} \cdot \frac{1}{2}\int_{-R}^{R} (\sqrt{r^2 - x^2})^2 dx$$

$$= \frac{1}{\pi r^2} \int_{-R}^{R} \left(\sqrt{r^2 - x^2}\right)^2 dx \longrightarrow f(x)$$

$$\text{even} = \frac{2}{\pi r^2} \int_{0}^{R} \left(\sqrt{r^2 - x^2}\right)^2 dx$$

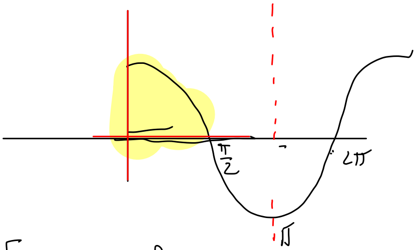
$$= \frac{2}{\pi r^2} \left[r^2 x - \frac{x^3}{3} \right]_0^{r}$$

$$= \frac{2}{\pi r^2} \frac{2r^3}{3} = \frac{4r}{3\pi}$$

Thus center of mass at $\left(0, \frac{4r}{3\pi}\right)$.

Find the Centroid of the region bounded by the curve

$y = \cos x$, $y = 0$, $x = 0$, and $x = \frac{\pi}{2}$



Find area first

$$A = \int_0^{\frac{\pi}{2}} \cos x \, dx = \sin x \Big]_0^{\frac{\pi}{2}} = 1$$

↓

Use formula

$$\bar{x} = \frac{1}{A}\int_a^b x f(x)\, dx \qquad \bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2$$

$$= \int_0^{\frac{\pi}{2}} x \cos x \, dx$$

$$= x \sin x \Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x \, dx$$

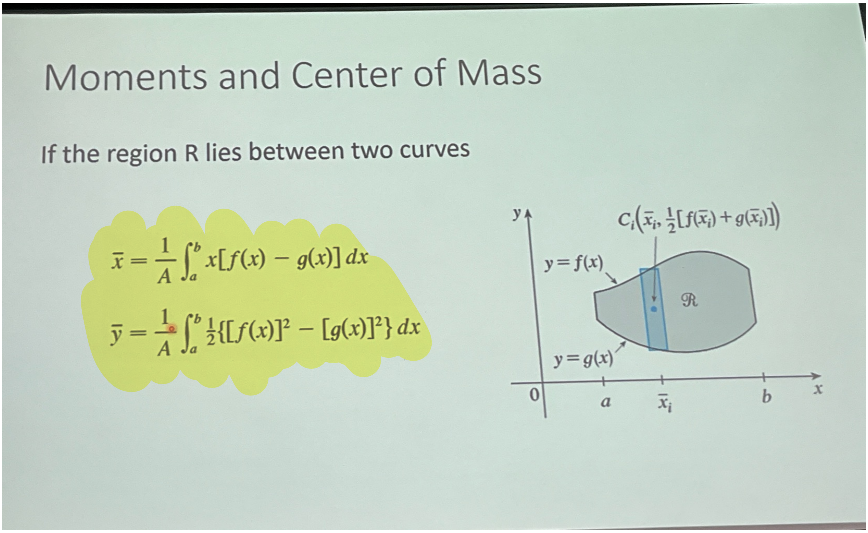
$$= \frac{\pi}{2} - 1$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^2 x \, dx$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} (1 + \cos 2x)\, dx = \frac{1}{4}\Big[x + \frac{1}{2}\sin 2x\Big]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{8}$$

Centroid at $\left(\frac{1}{2}\pi - 1, \frac{1}{8}\pi\right)$



Moments and Center of Mass

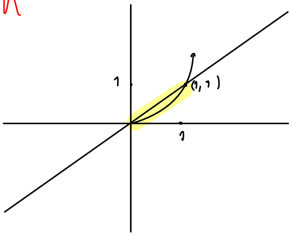
If the region R lies between two curves

$$\bar{x} = \frac{1}{A}\int_a^b x[f(x) - g(x)]\,dx$$

$$\bar{y} = \frac{1}{A}\int_a^b \tfrac{1}{2}\{[f(x)]^2 - [g(x)]^2\}\,dx$$

Find the region bounded by the line $y = x$ and parabola $y = x^2$

Centroid



$$\text{Area} = \int_0^1 x - x^2\,dx$$

$$= \left.\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$$\bar{x} = \frac{1}{\frac{1}{6}}\int_0^1 x\left[x - x^2\right]dx$$

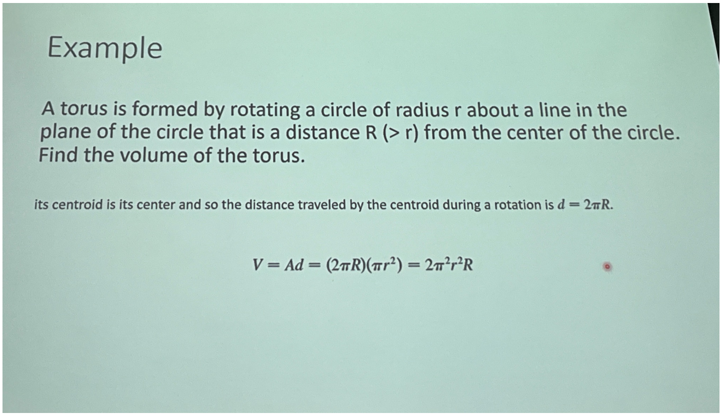
$$= 6\int_0^1 x^2 - x^3 = 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{2}$$

$$\bar{y} = \frac{1}{\frac{1}{6}} \times \frac{1}{2}\int_0^1 [x]^2 - [x^2]^2\,dx$$

$$= 3\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{2}{5}$$

Centroid at $\left(\frac{7}{2}, \frac{2}{5}\right)$

Theorem



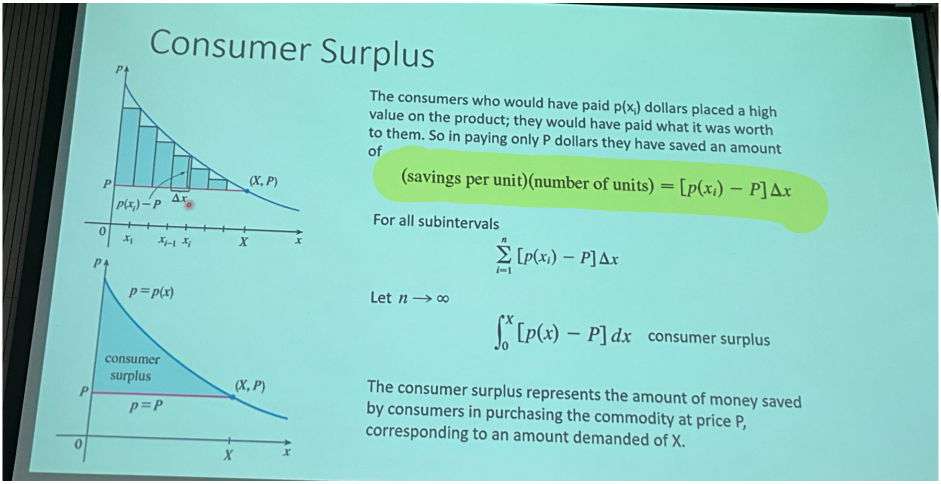
Example

A torus is formed by rotating a circle of radius r about a line in the plane of the circle that is a distance R (> r) from the center of the circle. Find the volume of the torus.

its centroid is its center and so the distance traveled by the centroid during a rotation is $d = 2\pi R$.

$$V = Ad = (2\pi R)(\pi r^2) = 2\pi^2 r^2 R$$

Consumer Surplus



Consumer Surplus

The consumers who would have paid $p(x_i)$ dollars placed a high value on the product; they would have paid what it was worth to them. So in paying only P dollars they have saved an amount of

(savings per unit)(number of units) $= [p(x_i) - P]\Delta x$

For all subintervals

$$\sum_{i=1}^{n} [p(x_i) - P]\Delta x$$

Let $n \to \infty$

$$\int_0^X [p(x) - P]\,dx \quad \text{consumer surplus}$$

The consumer surplus represents the amount of money saved by consumers in purchasing the commodity at price P, corresponding to an amount demanded of X.

The demand for a products in dollars, is $p = 1200 - 0.2x - 0.0001x^2$

Find consumer surpls when the sales level is 500

$$\text{So } p = 1200 - (0.2)(500) - (0.0001)(500)^2 = 1075$$

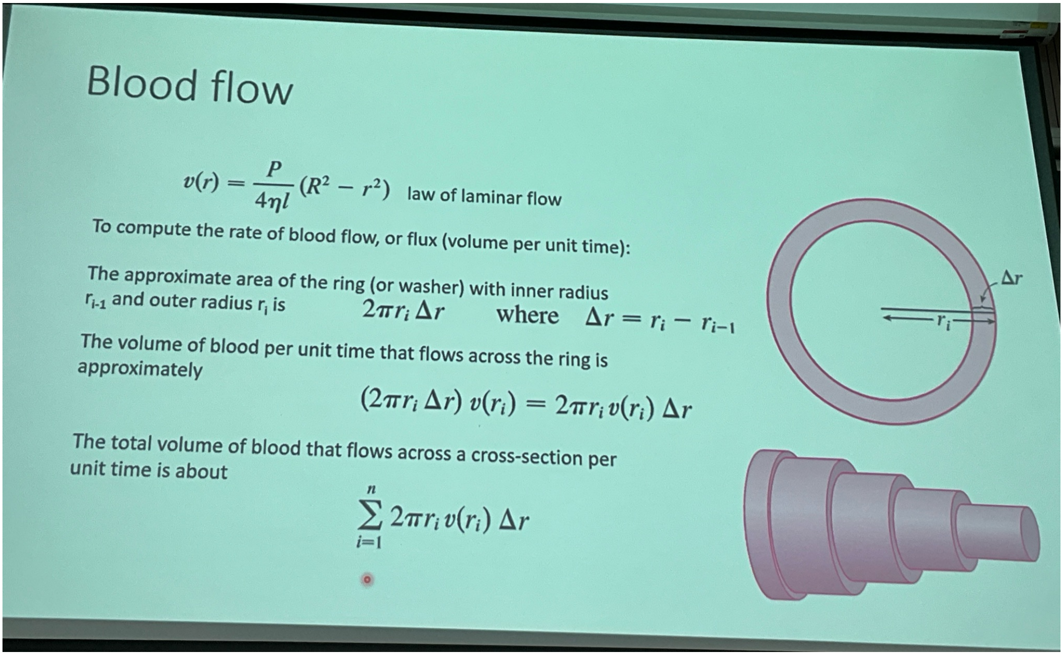
$$\text{Consumer surplus} = \int_0^{500} [p(x) - P]\,dx = \int_0^{500} (1200 - 0.2x - 0.0001x^2 - 1075)\,dx$$

$\downarrow$

07y

$$= (125)(500) - (0.1)(500)^2 - \frac{(0.0001)(500)^3}{3}$$

$= 33,333.33$ \$.



Blood flow

$$v(r) = \frac{P}{4\eta l}(R^2 - r^2)$$ law of laminar flow

To compute the rate of blood flow, or flux (volume per unit time):

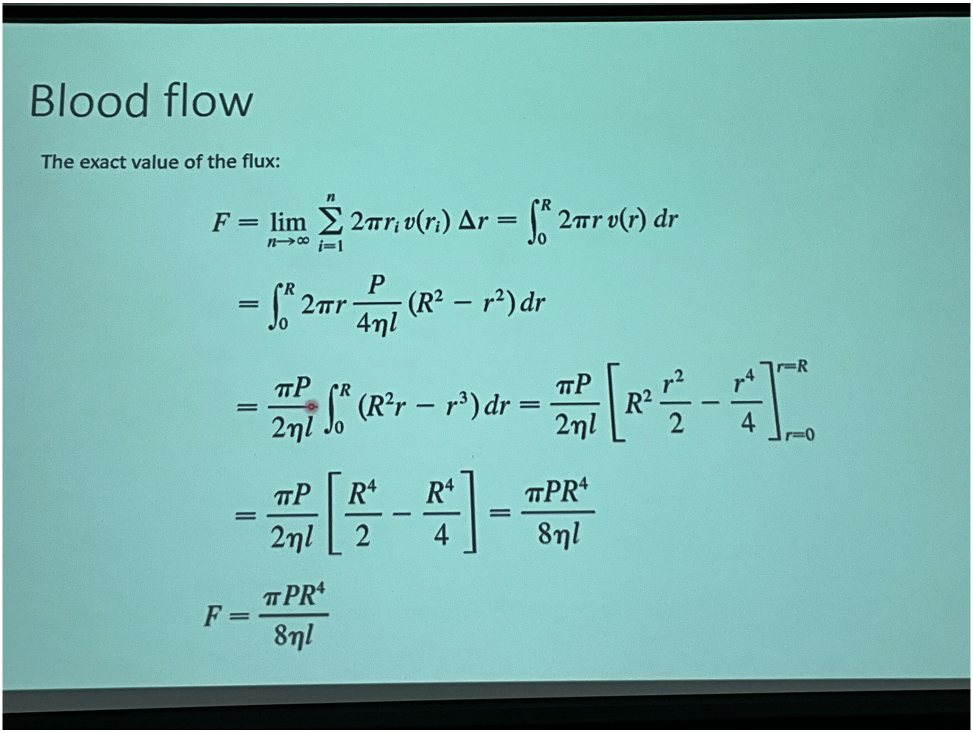
The approximate area of the ring (or washer) with inner radius r_{i-1} and outer radius r_i is $2\pi r_i \, \Delta r$ where $\Delta r = r_i - r_{i-1}$

The volume of blood per unit time that flows across the ring is approximately

$$(2\pi r_i \, \Delta r) \, v(r_i) = 2\pi r_i \, v(r_i) \, \Delta r$$

The total volume of blood that flows across a cross-section per unit time is about

$$\sum_{i=1}^{n} 2\pi r_i \, v(r_i) \, \Delta r$$



Blood flow

The exact value of the flux:

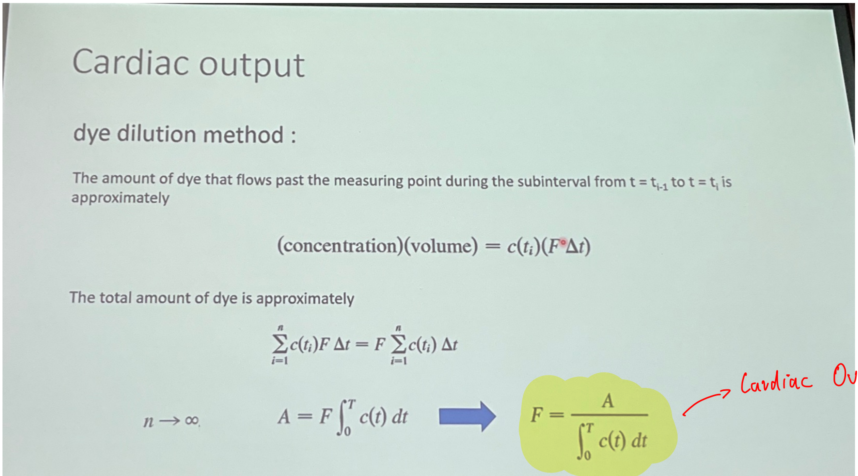
$$F = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi r_i \, v(r_i) \, \Delta r = \int_0^R 2\pi r \, v(r) \, dr$$

$$= \int_0^R 2\pi r \frac{P}{4\eta l}(R^2 - r^2) \, dr$$

$$= \frac{\pi P}{2\eta l} \int_0^R (R^2 r - r^3) \, dr = \frac{\pi P}{2\eta l}\left[R^2 \frac{r^2}{2} - \frac{r^4}{4}\right]_{r=0}^{r=R}$$

$$= \frac{\pi P}{2\eta l}\left[\frac{R^4}{2} - \frac{R^4}{4}\right] = \frac{\pi P R^4}{8\eta l}$$

$$F = \frac{\pi P R^4}{8\eta l}$$



Cardiac output

dye dilution method :

The amount of dye that flows past the measuring point during the subinterval from t = t_{i-1} to t = t_i is approximately

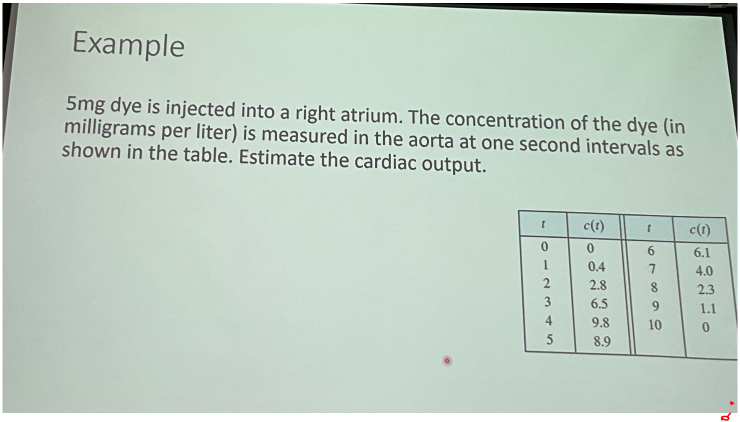
$$(\text{concentration})(\text{volume}) = c(t_i)(F\Delta t)$$

The total amount of dye is approximately

$$\sum_{i=1}^{n} c(t_i)F\,\Delta t = F\sum_{i=1}^{n} c(t_i)\,\Delta t$$

$$n \to \infty, \qquad A = F\int_0^T c(t)\,dt \quad \Rightarrow \quad F = \frac{A}{\int_0^T c(t)\,dt}$$

Cardiac Output formula



Example

5mg dye is injected into a right atrium. The concentration of the dye (in milligrams per liter) is measured in the aorta at one second intervals as shown in the table. Estimate the cardiac output.

t	$c(t)$	t	$c(t)$
0	0	6	6.1
1	0.4	7	4.0
2	2.8	8	2.3
3	6.5	9	1.1
4	9.8	10	0
5	8.9		

เริ่มตั้งแต่ t=0 ตลอด

$A = 5 \qquad n = 10 \qquad \Delta x = \frac{10-0}{10} = 1$

Simpson Rule = $\frac{1}{3}\Big[0.4(4) + 2(2.8) + \quad + 4(6.5) + 2(9.8) + 4(8.9) + 2(6.1)$

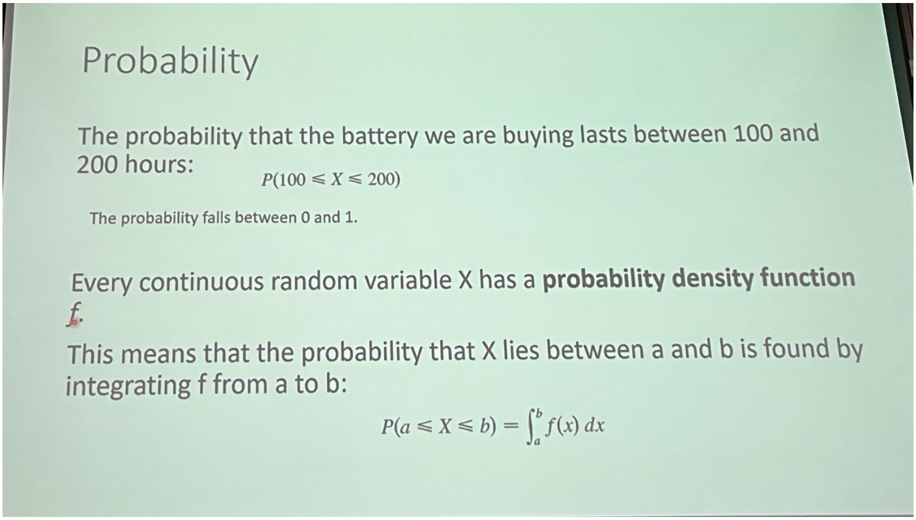
$+ 4(4.0) + 2(2.3) + 4(1.1)\Big]$

$= \int_0^{10} c(t) \approx 41.87$

5mg die injected

Formula for Cardiac output = $\frac{5}{41.87} \approx 0.12\,L/s = 7.2\,L/min$

Probability



Probability

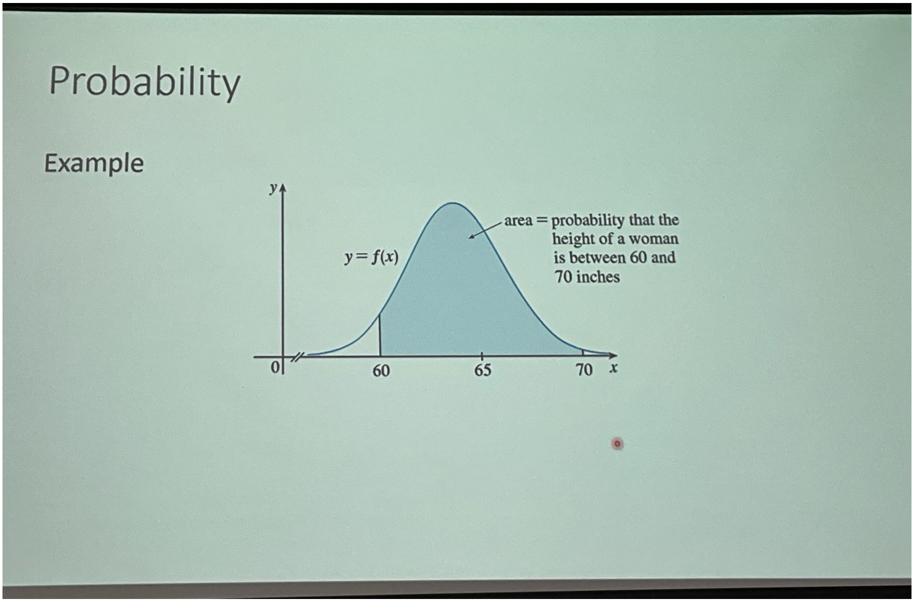
The probability that the battery we are buying lasts between 100 and 200 hours: $P(100 \leqslant X \leqslant 200)$

The probability falls between 0 and 1.

Every continuous random variable X has a **probability density function** *f*.

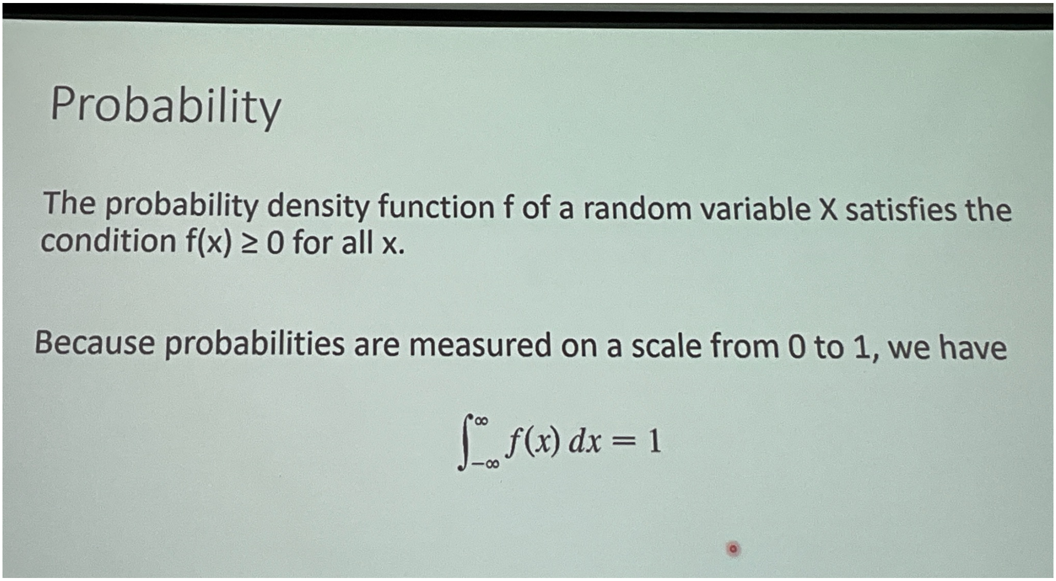
This means that the probability that X lies between a and b is found by integrating f from a to b:

$$P(a \leqslant X \leqslant b) = \int_a^b f(x)\,dx$$



Probability

Example

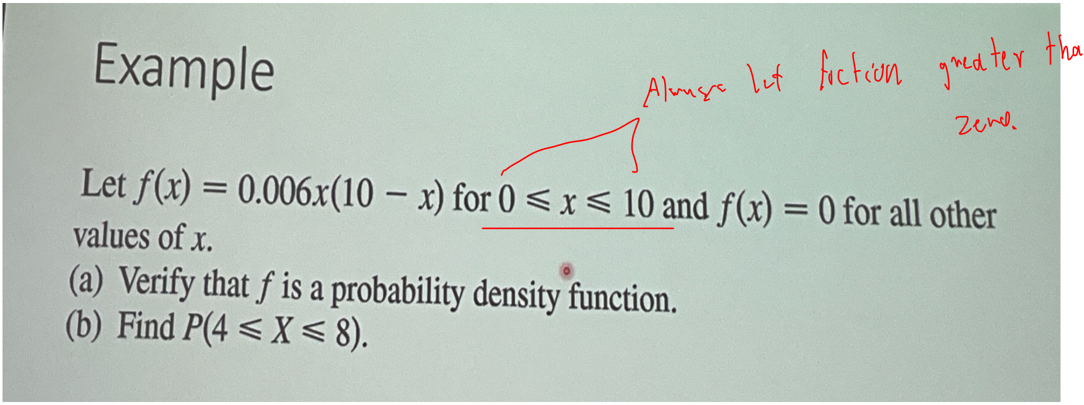


Probability

The probability density function f of a random variable X satisfies the condition f(x) ≥ 0 for all x.

Because probabilities are measured on a scale from 0 to 1, we have

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$



Example

Always let fuction greater than zero.

Let $f(x) = 0.006x(10 - x)$ for $0 \leqslant x \leqslant 10$ and $f(x) = 0$ for all other values of x.

(a) Verify that f is a probability density function.

(b) Find $P(4 \leqslant X \leqslant 8)$.

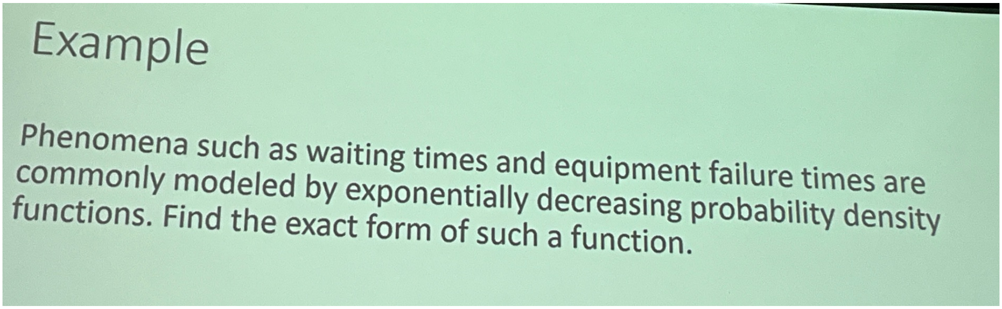
(a) = $\int_0^{10} 0.006x(10-x) =$

$= \int_0^{10} 0.06x - 0.006x^2$

$= \frac{0.06x^2}{2} - \frac{0.006x^3}{3} \Big|_0^{10} = 1$

#

b) = P(4 ≤ x ≤ 8) = $\int_4^8 f(x)\,dx \approx 0.549$



Example

Phenomena such as waiting times and equipment failure times are commonly modeled by exponentially decreasing probability density functions. Find the exact form of such a function.

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ Ae^{-ct} & \text{if } t \geq 0 \end{cases}$$

$t<0 = 0$

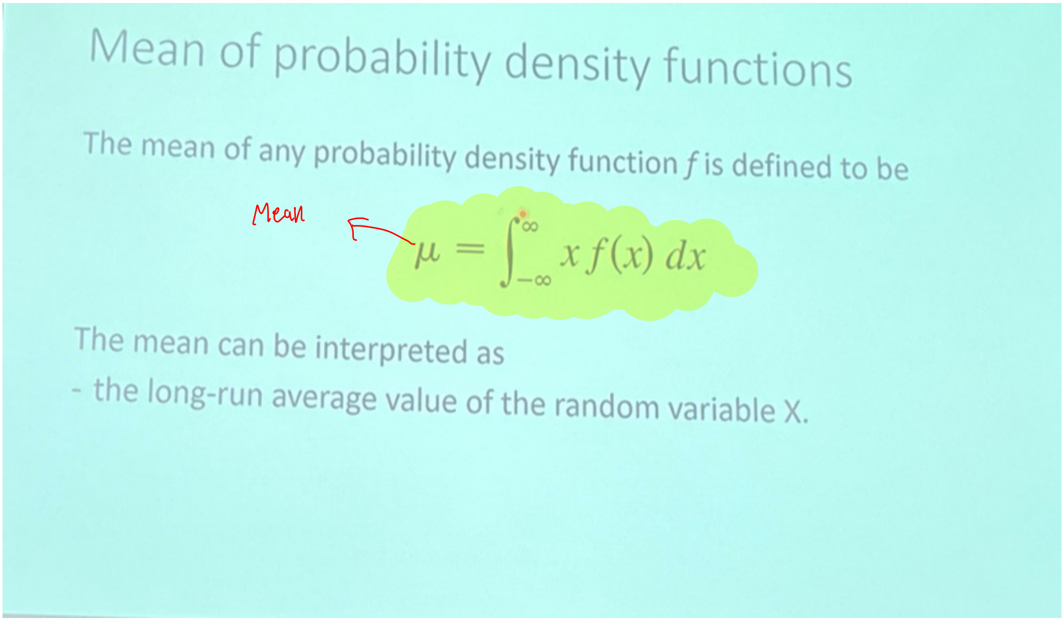
$$1 = \int_{-\infty}^{\infty} f(t)\,dt = \int_{-\infty}^{0} f(t)\,dt + \int_{0}^{\infty} f(t)\,dt$$

$$= \int_0^{\infty} Ae^{-ct}\,dt = \lim_{x\to\infty} \int_0^x Ae^{-ct}\,dt$$

$$= \lim_{x\to\infty} \left[-\frac{A}{C}e^{-ct}\right]\Bigg|_0^x = \lim_{x\to\infty} \frac{A}{C}\left(1-e^{-cx}\right)$$

$$= \frac{A}{C} \longrightarrow \frac{A}{C} = 1 \quad \text{so } A = C$$

So every exponential density function has form $f(t) = \begin{cases} 0, & \text{if } t<0 \\ Ce^{-ct} & \text{if } t \geq 0 \end{cases}$



Mean of probability density functions

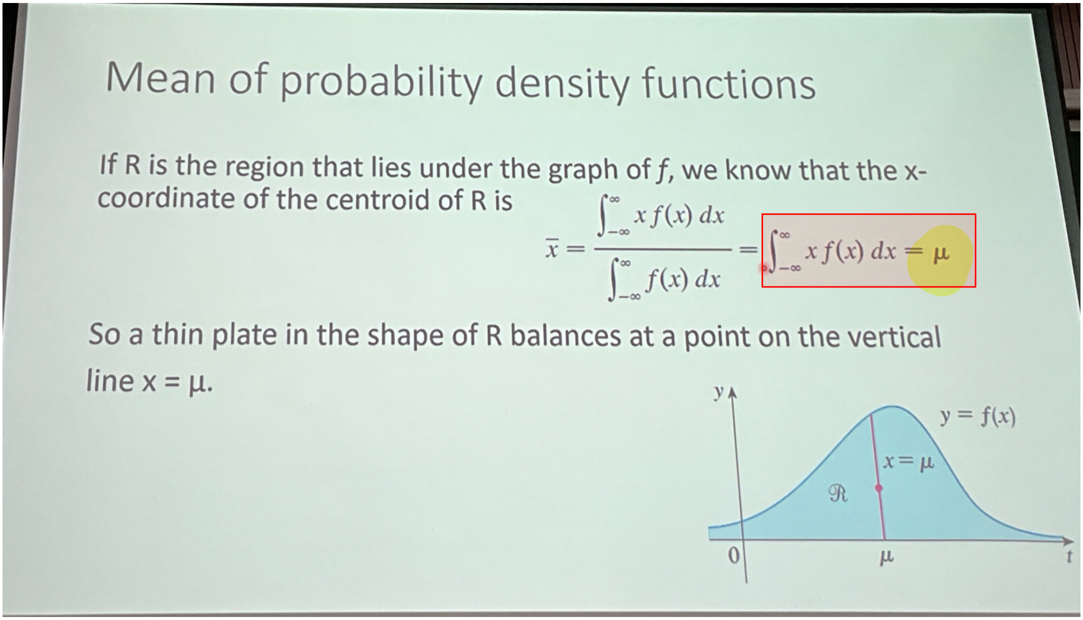
The mean of any probability density function *f* is defined to be

Mean

$$\mu = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

The mean can be interpreted as

- the long-run average value of the random variable X.



Mean of probability density functions

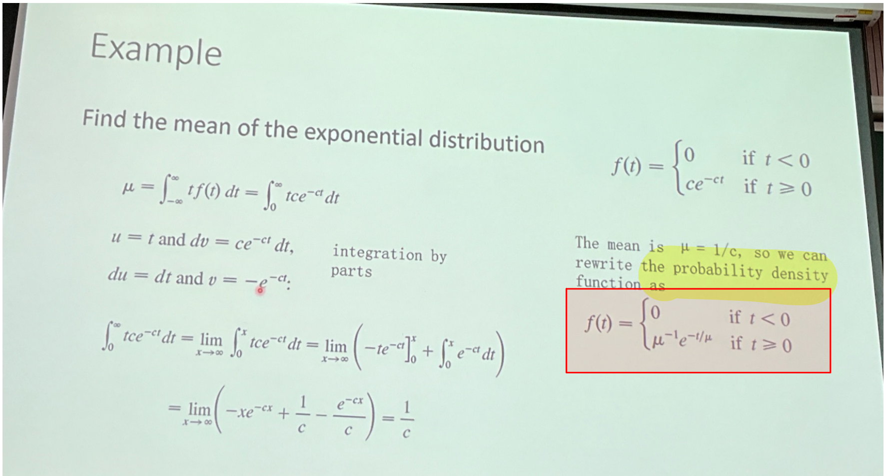
If R is the region that lies under the graph of *f*, we know that the x-coordinate of the centroid of R is

$$\bar{x} = \frac{\int_{-\infty}^{\infty} x\, f(x)\, dx}{\int_{-\infty}^{\infty} f(x)\, dx} = \boxed{\int_{-\infty}^{\infty} x\, f(x)\, dx = \mu}$$

So a thin plate in the shape of R balances at a point on the vertical line x = μ.

Find the mean of the exponential distribution

$$f(t) = \begin{cases} 0 & \text{if } t<0 \\ ce^{-ct} & \text{if } t \geq 0 \end{cases}$$



Example

Find the mean of the exponential distribution

$$\mu = \int_{-\infty}^{\infty} t f(t)\, dt = \int_0^{\infty} tce^{-ct}\, dt$$

$u = t$ and $dv = ce^{-ct}\, dt$,

$du = dt$ and $v = -e^{-ct}$.

integration by parts

$$\int_0^{\infty} tce^{-ct} dt = \lim_{x\to\infty} \int_0^x tce^{-ct} dt = \lim_{x\to\infty} \left(-te^{-ct}\Big]_0^x + \int_0^x e^{-ct} dt \right)$$

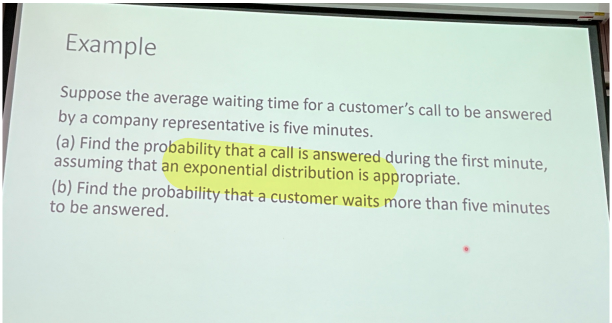
$$= \lim_{x\to\infty} \left(-xe^{-cx} + \frac{1}{c} - \frac{e^{-cx}}{c} \right) = \frac{1}{c}$$

$$f(t) = \begin{cases} 0 & \text{if } t<0 \\ ce^{-ct} & \text{if } t \geqslant 0 \end{cases}$$

The mean is $\mu = 1/c$, so we can rewrite the probability density function as

$$f(t) = \begin{cases} 0 & \text{if } t<0 \\ \mu^{-1}e^{-t/\mu} & \text{if } t \geqslant 0 \end{cases}$$

probability density function = $N^{-1}e^{-t/N}$ if $t \geq 0$



Example

Suppose the average waiting time for a customer's call to be answered by a company representative is five minutes.

(a) Find the probability that a call is answered during the first minute, assuming that an exponential distribution is appropriate.

(b) Find the probability that a customer waits more than five minutes to be answered.

จดใส่ cheat sheet.

Mean property density function. $= N^{-1} e^{-\frac{t}{N}}$ if $t \geq 0$

(a) find $N = \int_0^1 \frac{1}{5} e^{-\frac{t}{5}}$

$$= 0.2(-5)e^{-\frac{t}{5}} \Big|_0^1$$

$$= 1 - e^{-\frac{1}{5}} \approx 0.1813$$

↓

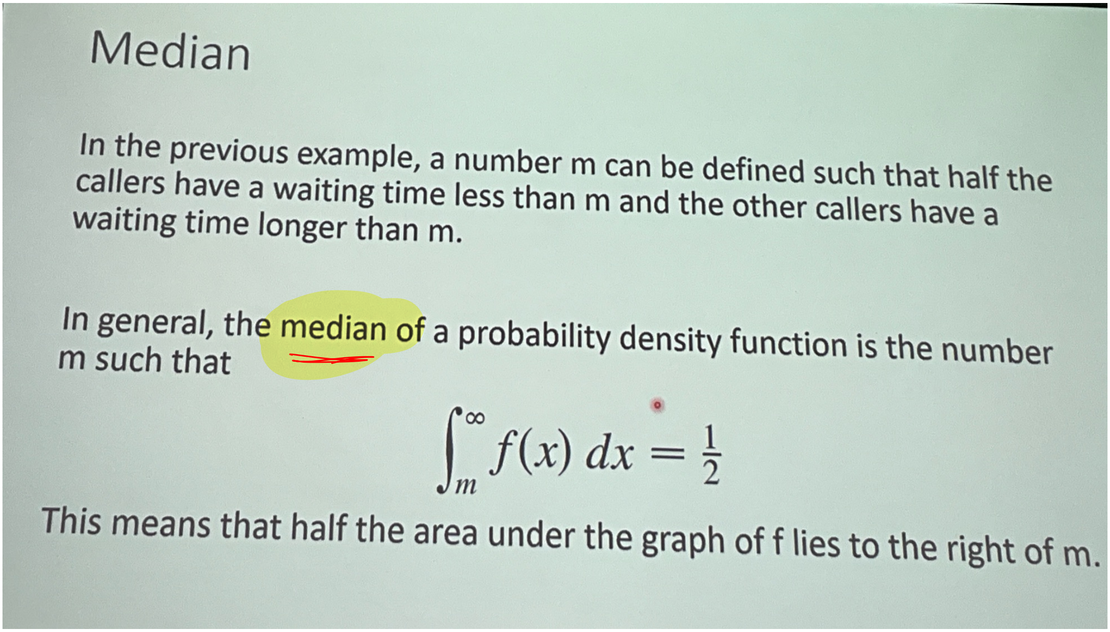
Probability the call answer = 18%.

(b) $P(T>5) = \int_5^\infty f(t)\,dt = \int_5^\infty 0.2\, e^{-\frac{t}{5}}\,dt$

$$= \lim_{x \to \infty} \int_5^x 0.2\, e^{-\frac{t}{5}}\,dt = \lim_{x \to \infty} \left(e^{-1} - e^{-\frac{x}{5}}\right)$$

$$= \frac{1}{e} - 0 \approx 0.368 \text{ or } 37\%$$

#



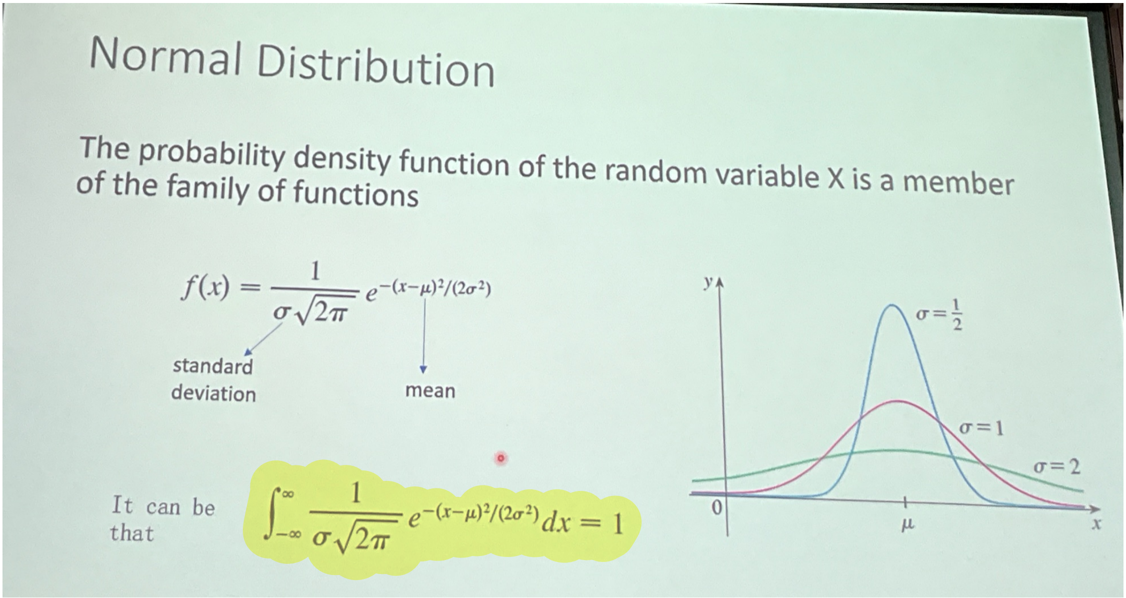
Median

In the previous example, a number m can be defined such that half the callers have a waiting time less than m and the other callers have a waiting time longer than m.

In general, the median of a probability density function is the number m such that

$$\int_{m}^{\infty} f(x)\,dx = \frac{1}{2}$$

This means that half the area under the graph of f lies to the right of m.



Normal Distribution

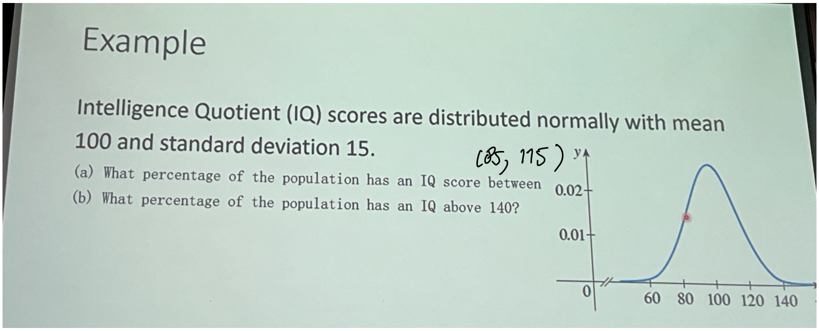
The probability density function of the random variable X is a member of the family of functions

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

standard deviation

mean

It can be that $$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}\,dx = 1$$



Example

Intelligence Quotient (IQ) scores are distributed normally with mean 100 and standard deviation 15.

(a) What percentage of the population has an IQ score between

(b) What percentage of the population has an IQ above 140?

(85, 115)

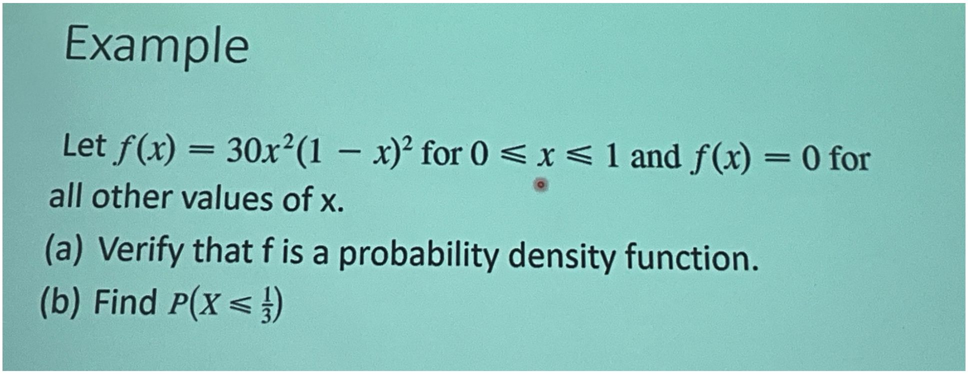
(a) $= \int_{85}^{115} \frac{1}{15\sqrt{2\pi}} \frac{e^{-(x-100)^2}}{2(15)^2} \, dx$

$= \approx 0.68$

(b) $P(X > 140) = \int_{140}^{\infty} \frac{1}{15\sqrt{2\pi}} e^{-(x-100)^2/450} \, dx$

$P(X > 140) \approx \int_{140}^{200} \frac{1}{15\sqrt{2\pi}} e^{-(x-100)^2/450} \, dx = 0.0038$

#



Example

Let $f(x) = 30x^2(1-x)^2$ for $0 \leqslant x \leqslant 1$ and $f(x) = 0$ for all other values of x.

(a) Verify that f is a probability density function.

(b) Find $P(X \leqslant \frac{1}{3})$

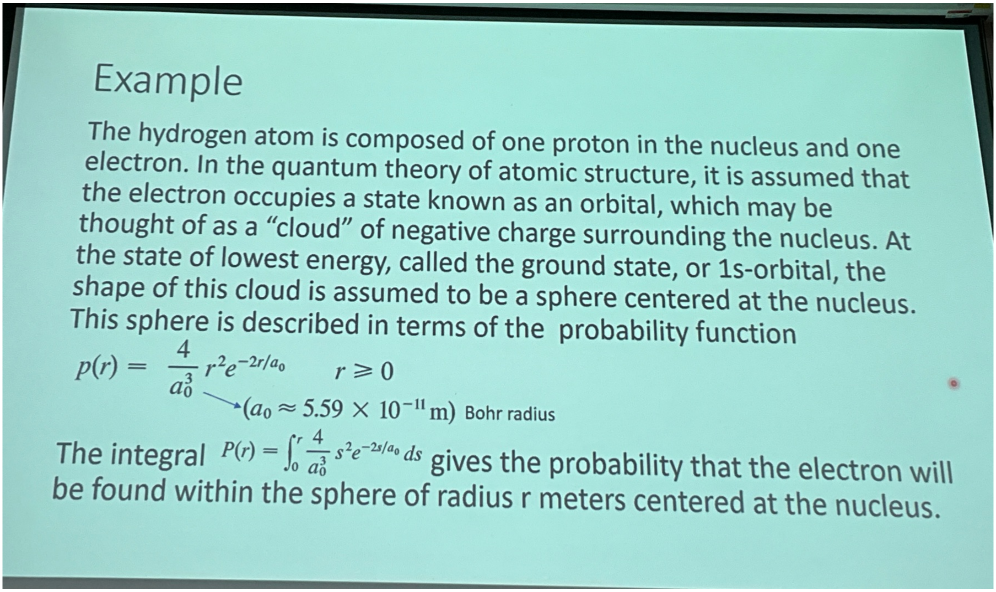
(a) : $\int_0^1 30x^2(1-x)^2 = 1$

$= 1$ so prove

for $0 \leq x \leq 1$

(b) Find $p(x \leq \frac{1}{3}) = \int_{-\infty}^{\frac{1}{3}} f(x)\,dx = \int_0^{\frac{1}{3}} 30x^2(1-x^2)\,dx.$

$= 0.346 \approx 34.6$ percent



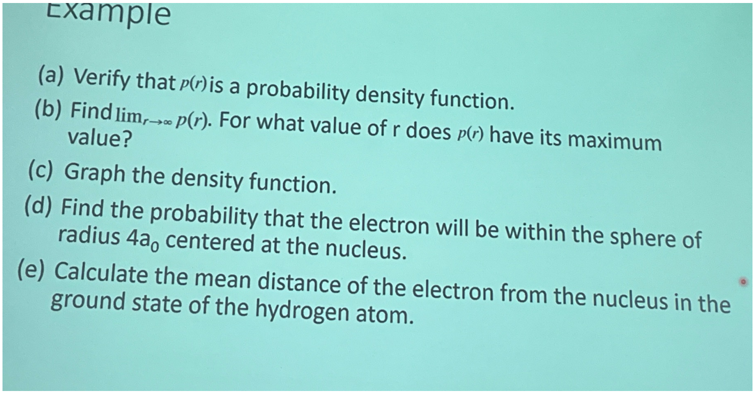
Example

The hydrogen atom is composed of one proton in the nucleus and one electron. In the quantum theory of atomic structure, it is assumed that the electron occupies a state known as an orbital, which may be thought of as a "cloud" of negative charge surrounding the nucleus. At the state of lowest energy, called the ground state, or 1s-orbital, the shape of this cloud is assumed to be a sphere centered at the nucleus. This sphere is described in terms of the probability function

$$p(r) = \frac{4}{a_0^3} r^2 e^{-2r/a_0} \qquad r \geqslant 0$$

$(a_0 \approx 5.59 \times 10^{-11}\,\text{m})$ Bohr radius

The integral $P(r) = \int_0^r \frac{4}{a_0^3} s^2 e^{-2s/a_0}\,ds$ gives the probability that the electron will be found within the sphere of radius r meters centered at the nucleus.



Example

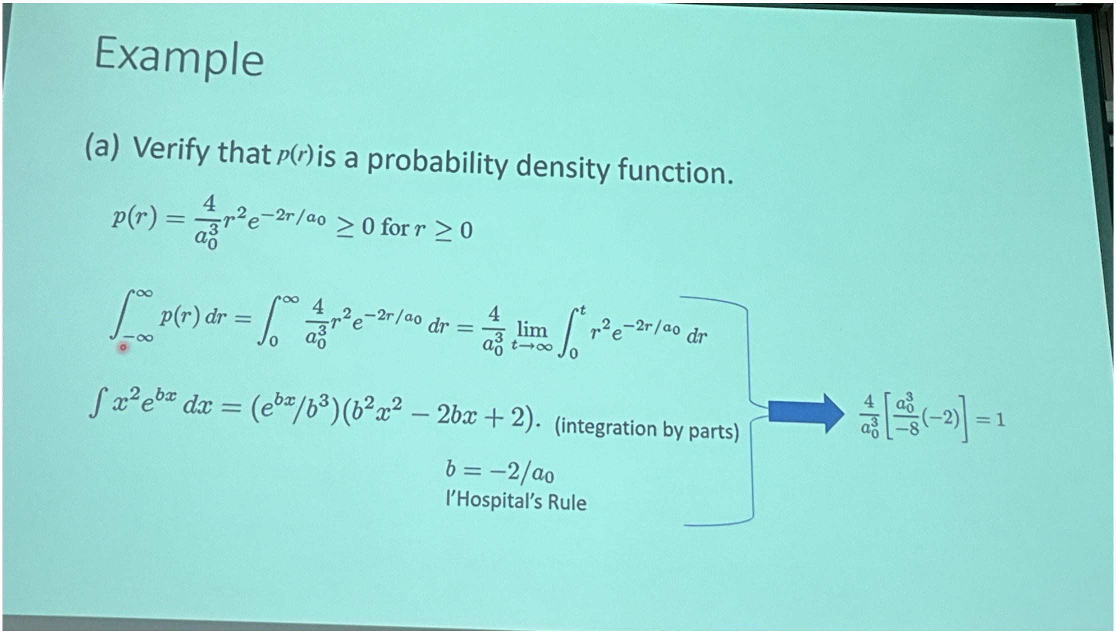
(a) Verify that $p(r)$ is a probability density function.

(b) Find $\lim_{r\to\infty} p(r)$. For what value of r does $p(r)$ have its maximum value?

(c) Graph the density function.

(d) Find the probability that the electron will be within the sphere of radius $4a_0$ centered at the nucleus.

(e) Calculate the mean distance of the electron from the nucleus in the ground state of the hydrogen atom.



Example

(a) Verify that $p(r)$ is a probability density function.

$$p(r) = \frac{4}{a_0^3} r^2 e^{-2r/a_0} \geq 0 \text{ for } r \geq 0$$

$$\int_{-\infty}^{\infty} p(r)\,dr = \int_0^{\infty} \frac{4}{a_0^3} r^2 e^{-2r/a_0}\,dr = \frac{4}{a_0^3} \lim_{t \to \infty} \int_0^t r^2 e^{-2r/a_0}\,dr$$

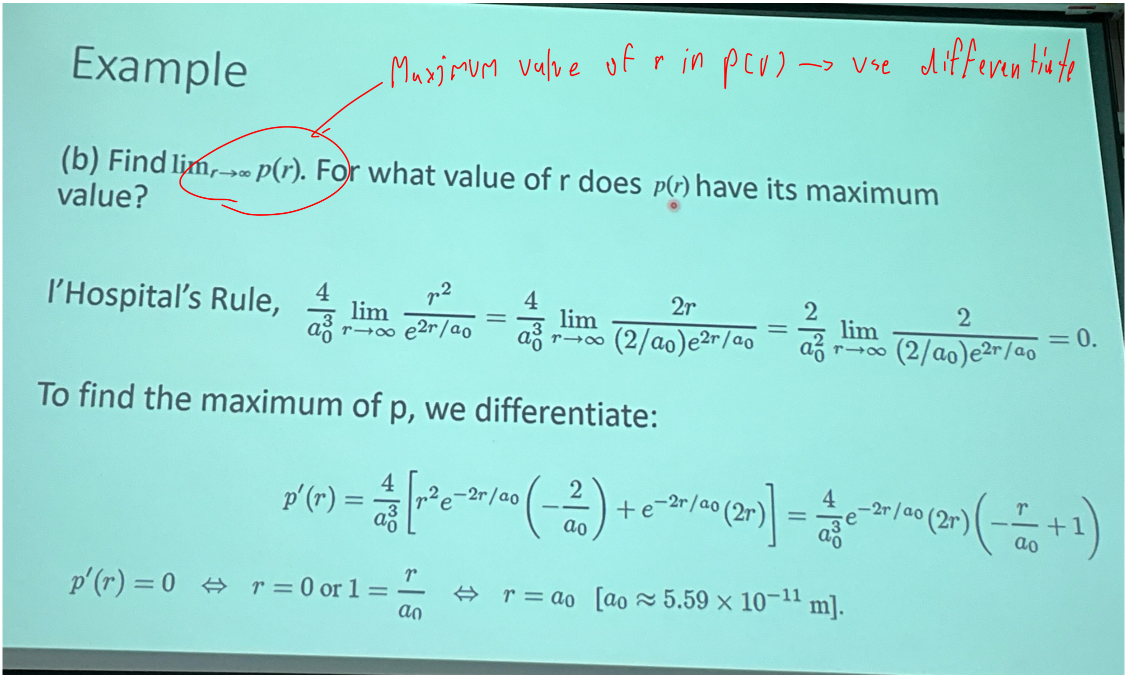
$$\int x^2 e^{bx}\,dx = (e^{bx}/b^3)(b^2x^2 - 2bx + 2).$$

(integration by parts)

$b = -2/a_0$

l'Hospital's Rule

$$\frac{4}{a_0^3}\left[\frac{a_0^3}{-8}(-2)\right] = 1$$



Example

Maximum value of r in p(r) -> use differentiate

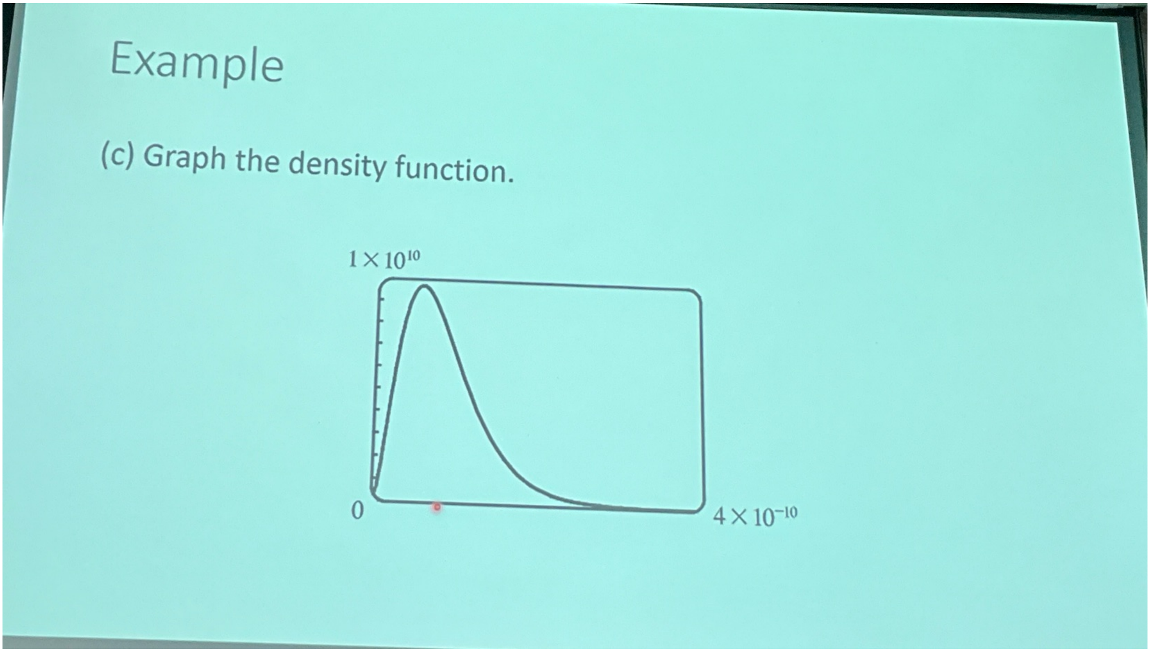
(b) Find $\lim_{r \to \infty} p(r)$. For what value of r does $p(r)$ have its maximum value?

l'Hospital's Rule, $\frac{4}{a_0^3} \lim_{r \to \infty} \frac{r^2}{e^{2r/a_0}} = \frac{4}{a_0^3} \lim_{r \to \infty} \frac{2r}{(2/a_0)e^{2r/a_0}} = \frac{2}{a_0^2} \lim_{r \to \infty} \frac{2}{(2/a_0)e^{2r/a_0}} = 0.$

To find the maximum of p, we differentiate:

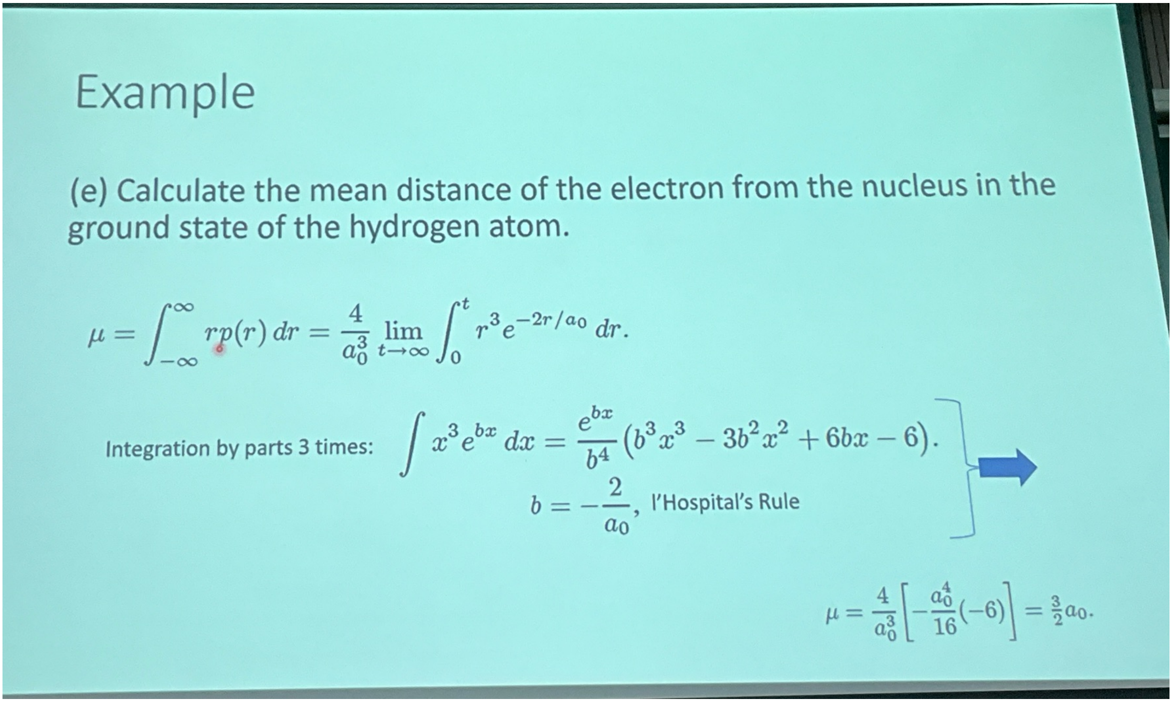
$$p'(r) = \frac{4}{a_0^3}\left[r^2 e^{-2r/a_0}\left(-\frac{2}{a_0}\right) + e^{-2r/a_0}(2r)\right] = \frac{4}{a_0^3} e^{-2r/a_0}(2r)\left(-\frac{r}{a_0} + 1\right)$$

$$p'(r) = 0 \quad \Leftrightarrow \quad r = 0 \text{ or } 1 = \frac{r}{a_0} \quad \Leftrightarrow \quad r = a_0 \quad [a_0 \approx 5.59 \times 10^{-11} \text{ m}].$$



Example

(c) Graph the density function.



Example

(e) Calculate the mean distance of the electron from the nucleus in the ground state of the hydrogen atom.

$$\mu = \int_{-\infty}^{\infty} rp(r)\,dr = \frac{4}{a_0^3}\lim_{t\to\infty}\int_0^t r^3 e^{-2r/a_0}\,dr.$$

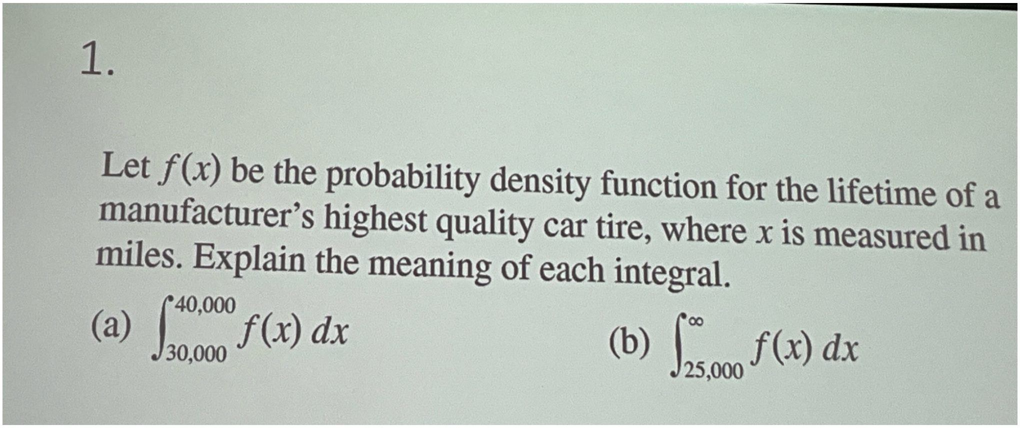
Integration by parts 3 times:

$$\int x^3 e^{bx}\,dx = \frac{e^{bx}}{b^4}\left(b^3x^3 - 3b^2x^2 + 6bx - 6\right).$$

$$b = -\frac{2}{a_0},$$ l'Hospital's Rule

$$\mu = \frac{4}{a_0^3}\left[-\frac{a_0^4}{16}(-6)\right] = \tfrac{3}{2}a_0.$$

Examples



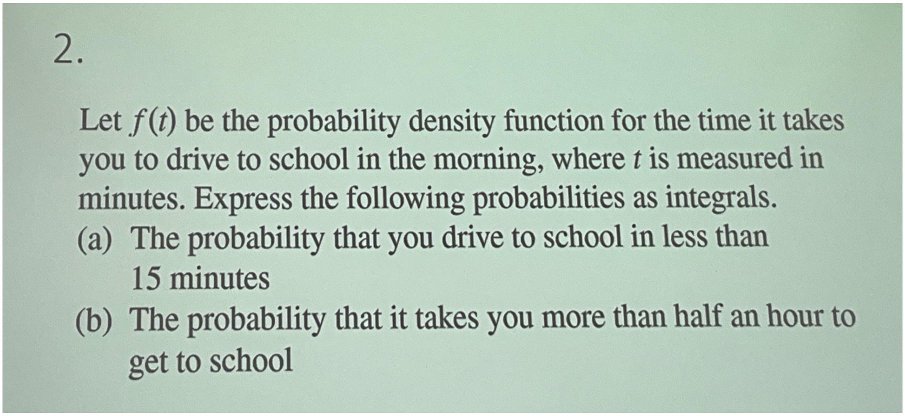
1.

Let $f(x)$ be the probability density function for the lifetime of a manufacturer's highest quality car tire, where x is measured in miles. Explain the meaning of each integral.

(a) $\int_{30,000}^{40,000} f(x)\,dx$ (b) $\int_{25,000}^{\infty} f(x)\,dx$

(a) The probability of Choosing a random Tire with life time between 30,000 and 40,000 miles.

(b) Same → higher than 2500 miles



2.

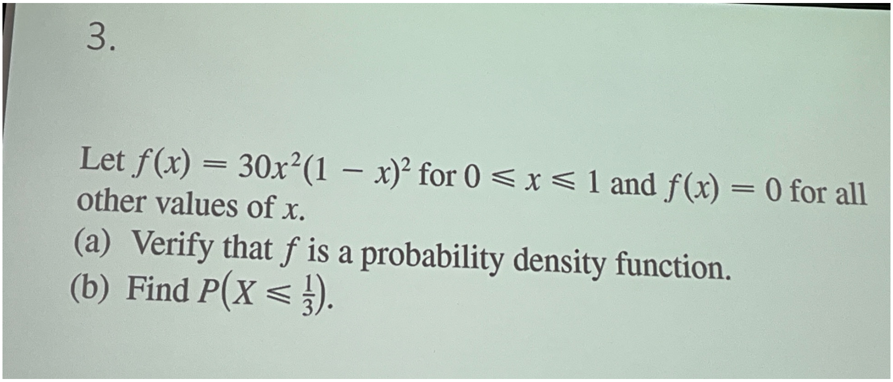
Let $f(t)$ be the probability density function for the time it takes you to drive to school in the morning, where t is measured in minutes. Express the following probabilities as integrals.

(a) The probability that you drive to school in less than 15 minutes

(b) The probability that it takes you more than half an hour to get to school

(a) $\int_{0}^{15} f(t)\ dt$

(b) $\int_{30}^{\infty} f(t)\ dt$



3.

Let $f(x) = 30x^2(1-x)^2$ for $0 \leqslant x \leqslant 1$ and $f(x) = 0$ for all other values of x.

(a) Verify that f is a probability density function.

(b) Find $P\left(X \leqslant \frac{1}{3}\right)$.

(a) $= \int_{0}^{1} 30x^2(1-x)^2 = \int_{0}^{1} 30x^2(x^2-2x+1)$

$= \int_{0}^{1} 30x^4 - 60x^3 + 30x^2$

$= 30\int_{0}^{1} x^4 - 2x^3 + x^2$

$30x^2 \geq 0 \longleftrightarrow (1-x)^2 \geq 0$

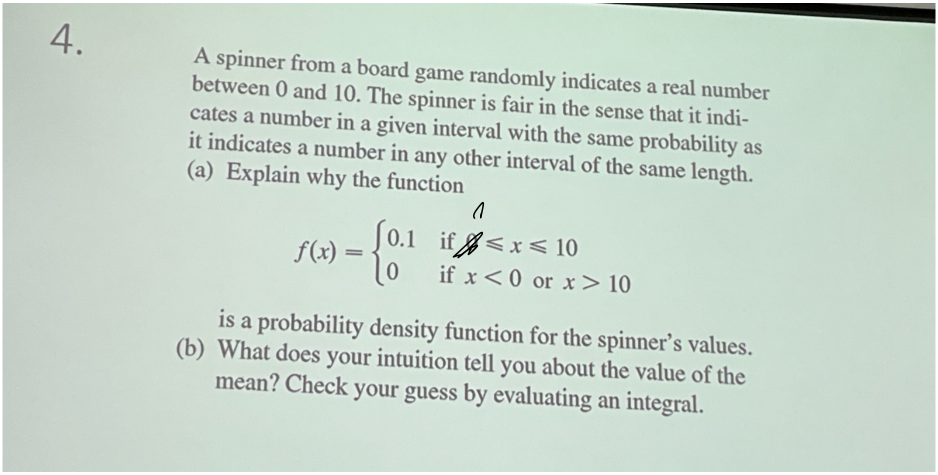
$$= 30\left[\frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3}\right]_0^1$$

$$= 30\left[\frac{1}{5} - \frac{1}{2} + \frac{1}{3}\right] = 1$$ #

(ii) $$30\left[\frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3}\right]_0^{\frac{1}{3}} = 0.207$$

$$\downarrow$$
$$0$$

$$= \int_{-\infty}^{\frac{1}{3}} f(x) \longrightarrow \int_{0}^{\frac{1}{3}} f(x)$$



4.

A spinner from a board game randomly indicates a real number between 0 and 10. The spinner is fair in the sense that it indicates a number in a given interval with the same probability as it indicates a number in any other interval of the same length.

(a) Explain why the function

$$f(x) = \begin{cases} 0.1 & \text{if } \overset{1}{\cancel{0}} \leqslant x \leqslant 10 \\ 0 & \text{if } x < 0 \text{ or } x > 10 \end{cases}$$

is a probability density function for the spinner's values.

(b) What does your intuition tell you about the value of the mean? Check your guess by evaluating an integral.

(a) $f(x) \geq 0$ ✓

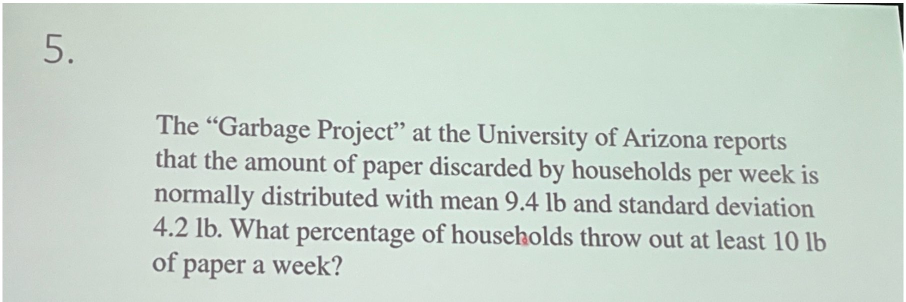
$$\int_{-\infty}^{\infty} f(x) = \int_{0}^{10} 0.1 = 0.1x\Big|_0^{10} = 1$$ ✓

(b)

$$N = \int_{-\infty}^{\infty} x f(x) = \int_{0}^{10} 0.1\, x = \left.\frac{0.1 x^2}{2}\right|_{0}^{10}$$

$$= 5$$

$\#$



5.

The "Garbage Project" at the University of Arizona reports that the amount of paper discarded by households per week is normally distributed with mean 9.4 lb and standard deviation 4.2 lb. What percentage of households throw out at least 10 lb of paper a week?

$p(x \geq 10)$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-N)^2/(2\sigma^2)}\, dx = 1$$

$$= \int_{10}^{\infty} \frac{1}{4.2\sqrt{2\pi}} \cdot e^{-(X-9.4)^2/(2(4.2)^2)}\, dx = 1$$

$$= \int_{10}^{X} \frac{1}{4.2\sqrt{2\pi}} \cdot e^{-(X-9.4)^2/(2(4.2)^2)}\, dx$$

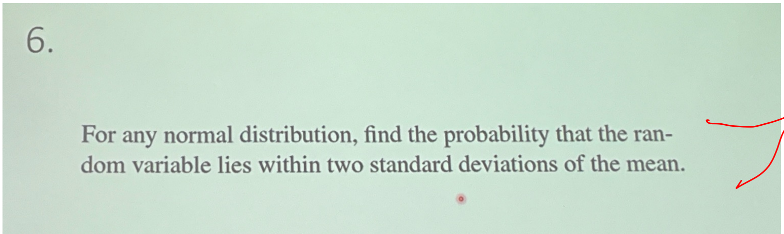
$$\downarrow$$

$$= \int_{70}^{100} \frac{1}{4.2\sqrt{2\pi}} \cdot e^{-(x-9.4)^2/(2(4.2)^2)}\, dx$$

$$\downarrow$$

$$\approx 0.443$$

✗



6.

For any normal distribution, find the probability that the random variable lies within two standard deviations of the mean.

Trick question

$$P(N-2S \le x \le N+2S)$$

$$= \int_{N-2S}^{N+2S} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-N)^2/(2S^2)}\, dx = 1$$

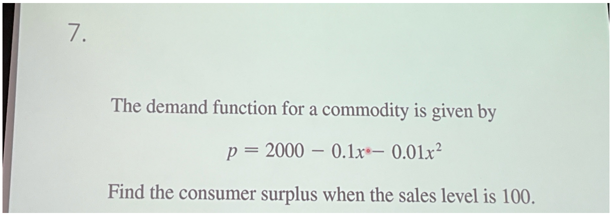
→ แทนเข้า x

$$\text{sub} \quad t = \frac{x-N}{S} \longrightarrow dt = \frac{1}{S}\, dx$$

$$= \int_{-2}^{2} \frac{1}{S\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right)(S\, dt) \approx 0.9545$$

✗

not calculate in calculator not correct.



7.

The demand function for a commodity is given by

$$p = 2000 - 0.1x - 0.01x^2$$

Find the consumer surplus when the sales level is 100.

$$= \int_0^{100} (p(x) - p)\, dx$$

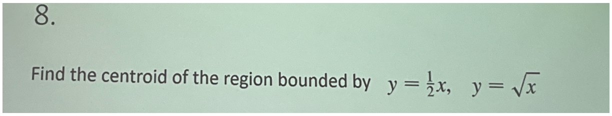
$$= p = 2000 - 0.1(100) - 0.01(100)^2$$

$$p = 1890$$

$$= \int_0^{100} (2000 - 0.1x - 0.01x^2 - 1890)\, dx$$

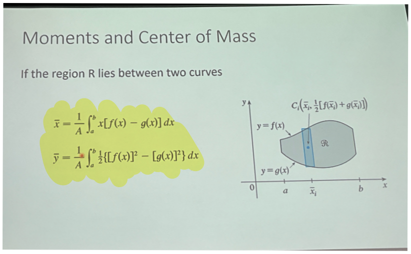
$$= \int_0^{100} 110 - 0.1x - 0.01x^2$$

$$\approx 7166.666\ \$$$



8.

Find the centroid of the region bounded by $y = \frac{1}{2}x, \quad y = \sqrt{x}$

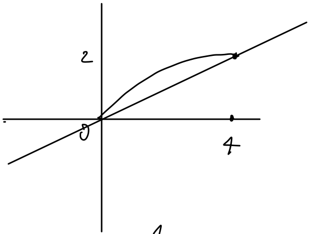


Moments and Center of Mass

If the region R lies between two curves

$$\bar{x} = \frac{1}{A}\int_a^b x[f(x) - g(x)]\, dx$$

$$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}\{[f(x)]^2 - [g(x)]^2\}\, dx$$



$\frac{1}{2}x = \sqrt{x}$ $\quad y^2 = 2\sqrt{x}^{\,2}$ $\quad x^2 - 4y = 0$ $\quad x(x-4) = 0$

$x^2 = 4x$ $\quad x = 4, 0$

$$\text{Area} = \int_0^4 \sqrt{x} - \frac{1}{2}x\, dx = \left.\frac{2x^{\frac{3}{2}}}{3} - \frac{x^2}{4}\right|_0^4 = \frac{4}{3}$$

$$\bar{x} = \frac{1}{\frac{4}{3}} \int_0^4 x\left[\sqrt{x} - \frac{1}{2}x\right] dx$$

$$= \frac{3}{4}\int_0^4 x^{\frac{3}{2}} - \frac{1}{2}x^2 \quad dx$$

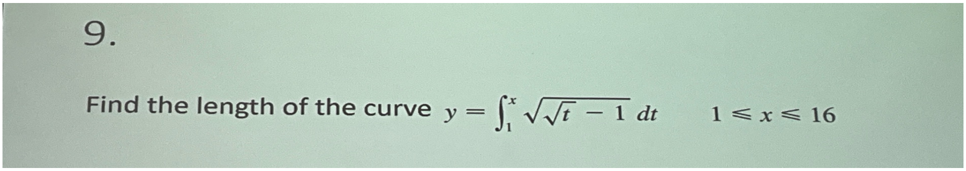
$$= \frac{3}{4}\left[\frac{32}{15}\right] = \frac{8}{5}$$

$$\bar{y} = \frac{1}{\frac{4}{3}}\int_0^4 \frac{1}{2}\left[(\sqrt{x})^2 - \left(\frac{1}{2}x\right)^2\right]$$

$$= \frac{3}{8}\int_0^4 x - \frac{1}{4}x^2 \quad dx$$

$$= \frac{3}{8}\left[\frac{8}{3}\right] = 1$$

centroid $= \left[\frac{8}{5}, 1\right]$



9.

Find the length of the curve $y = \int_1^x \sqrt{\sqrt{t} - 1}\, dt \qquad 1 \leqslant x \leqslant 16$

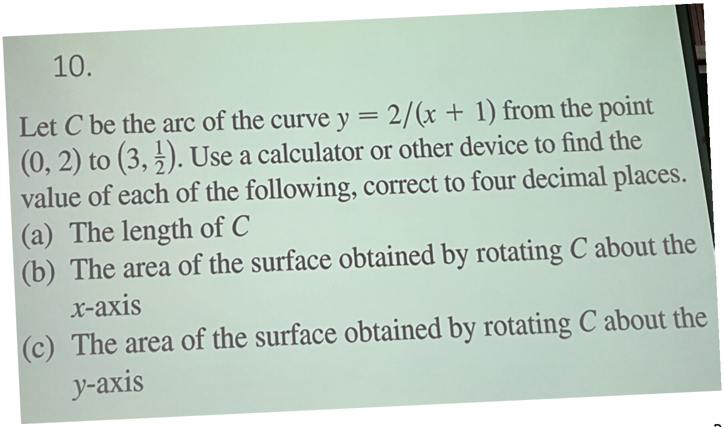
$$f(t) = \int_1^{x} (\sqrt{t} - 1)^{\frac{1}{2}}\, dt \qquad f'(t) = \sqrt{\sqrt{x} - 1}$$

Fundamental Theorem of Calculus

$$\downarrow$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + (\sqrt{x} - 1) = \sqrt{x}$$

$$L = \int_1^{16} \sqrt{\sqrt{x}}\, dx = \int_1^{16} x^{\frac{1}{4}} = \frac{4x^{\frac{5}{4}}}{5}\Bigg|_1^{16} = \frac{124}{5}$$



10.

Let C be the arc of the curve $y = 2/(x + 1)$ from the point $(0, 2)$ to $(3, \frac{1}{2})$. Use a calculator or other device to find the value of each of the following, correct to four decimal places.

(a) The length of C

(b) The area of the surface obtained by rotating C about the x-axis

(c) The area of the surface obtained by rotating C about the y-axis

$$(a) = C = \int_0^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}, \qquad \frac{dy}{dx} = \frac{(x+1)(0) - 2}{(x+1)^2} = -\frac{2}{(x+1)^2}$$

$$= \int_0^3 \sqrt{1 + \left(-\frac{2}{(x+1)^2}\right)^2} = \int_0^3 \sqrt{1 + \frac{4}{(x+1)^4}}$$

$$\downarrow$$

$$= \int_0^3 \sqrt{\frac{(x+1)^4+4}{(x+1)^4}}$$

length of $c \approx 3.5127$

(b) $S = \int_0^3 2\pi y\, ds = 2\pi \int_0^3 \frac{2}{x+1}\sqrt{1+\frac{4}{(x+1)^4}}\, dx \approx 22.1321$

(c) $S = \int_0^3 2\pi x\, ds = 2\pi \int_0^3 y\sqrt{1+\left(\frac{-2}{y^2}\right)^2} = 2\pi \int_0^3 \left(\frac{2}{y}-1\right)\sqrt{1+\frac{4}{y^4}}$

$y = \frac{2}{x+1}$

$y(x+1) = 2$

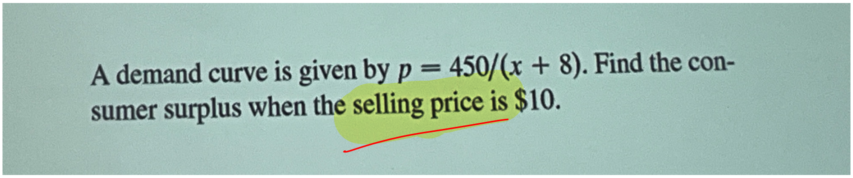
$x+1$

$x = \frac{2}{y} - 1$

$f'(y) = -2y^{-2}$

Easy way = y axis use x

$$\int_0^3 2\pi x\, ds = 2\pi \int_0^3 x\sqrt{1+\frac{4}{(x+1)^4}}\, dx = 29.8522$$



A demand curve is given by $p = 450/(x + 8)$. Find the consumer surplus when the selling price is \$10.

!!!

$= \int_a^b p(x) - \boxed{p}$ → sale price → Consumer Surplus

$= \int_a^{b \to \text{sale level}} p(x) - p$

$= p = (10) =$ ↗ sale level

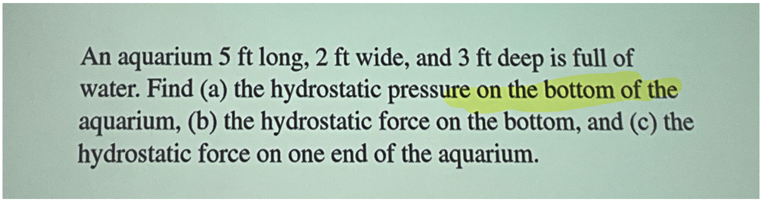
selling price $= p(x) = 10 \rightarrow \frac{450}{x+8} = 10 = 37 = x$

$=$

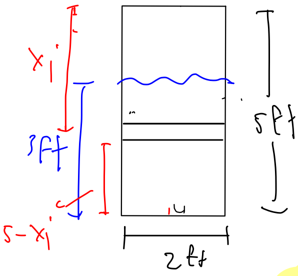
Cons surplus $= \int_0^{37} [p(x) - p]\, dx = \int_0^{37} [p(x) - 10]\, dx$

$= \int_0^{37} \left(\frac{450}{x+8} - 10 \right) dx$

$= 450 \ln(x+8) - 10x \Big|_0^{37} \approx \$ 907{,}25$



An aquarium 5 ft long, 2 ft wide, and 3 ft deep is full of water. Find (a) the hydrostatic pressure on the bottom of the aquarium, (b) the hydrostatic force on the bottom, and (c) the hydrostatic force on one end of the aquarium.



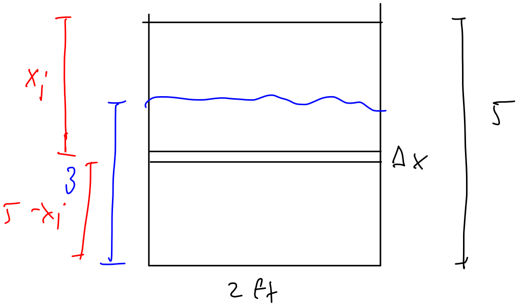
(a) = Pressure = $P = \rho g d$ → no need to evaludte integral

$= 62.5(3) = 187.5 \frac{lb}{ft^2}$

(b) $F = P \cdot A = (187.5)(A) = (187.5)(5 \times 2) = 1875$ lb → x ทั้ง aquarium

on the bottom so no need integral

(c)



$$\int_2^5 3-(5-x_i)$$

$$(x_i-2)\,2\,\Delta x$$

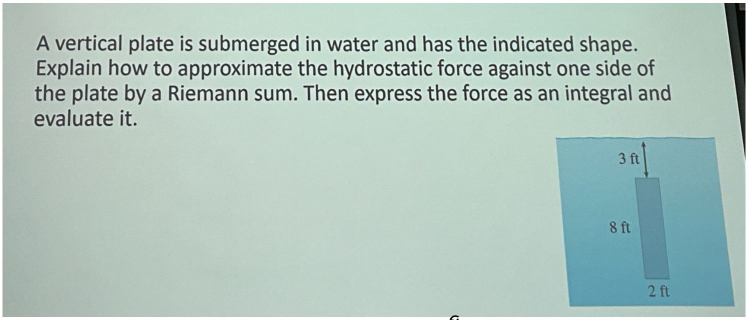
$$= 62.5\int_2^5 2x-4\;dx$$

$$\approx 562.5$$

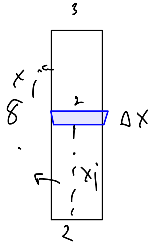
เริ่มจากข้างบน

area of strip = $2\Delta x$

Pressure $= 62.5\int_0^3 x)\,2\,dx \approx 562.5$ lb



A vertical plate is submerged in water and has the indicated shape. Explain how to approximate the hydrostatic force against one side of the plate by a Riemann sum. Then express the force as an integral and evaluate it.



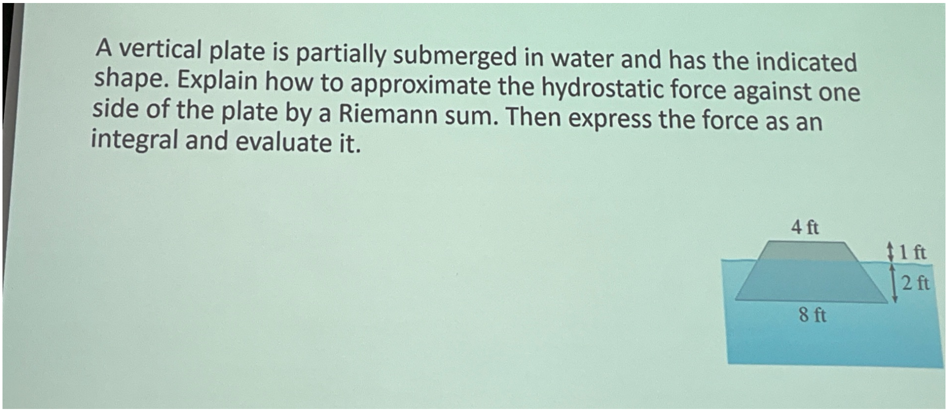
$$\approx \sum_{n=1}^{\infty} S x_j \cdot 2 \Delta x$$

$$= 62,5 \int_0^8 (11 - x_i) \cdot 2 \Delta x$$

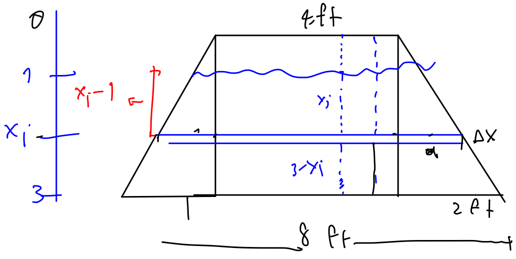
เริ่มที่ coordinate ของสิ่ง
ของ
จบที่สิ่งของ

$$= 62,5 \int_3^{11} x_i \; 2 \; dx$$

$$62.5 \int_0^{8} 2x \; dx \approx 7000 \text{ lb}$$



A vertical plate is partially submerged in water and has the indicated shape. Explain how to approximate the hydrostatic force against one side of the plate by a Riemann sum. Then express the force as an integral and evaluate it.



Area = Area = $4 + 2\left(\frac{2x}{3}\right)$

$\frac{a}{x_i} = \frac{2}{3}$

$a = \frac{2x}{3}$

↓

$a = 4 + \frac{4x}{3}$

Pressure on strip = $2 - (3 - x_i) = -1 + x_i$

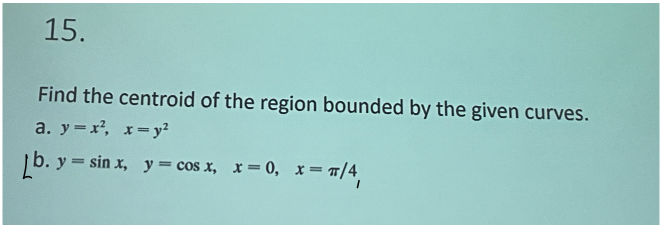
or $(x_i - 1)$

$= 62.5 \int_1^3 (x-1)\left(4 + \frac{4x}{3}\right) dx$

$\approx 888.8\overline{88}$

Ans = $888.8\overline{8}$

ft



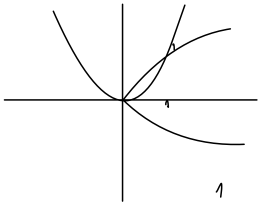
15.

Find the centroid of the region bounded by the given curves.

a. $y = x^2$, $x = y^2$

b. $y = \sin x$, $y = \cos x$, $x = 0$, $x = \pi/4$

a.



$x^2 = \sqrt{x}$

$x^2 - \sqrt{x} = 0$

$\sqrt{x}(x^{\frac{3}{2}} - 1) =$

$(x^{\frac{3}{2}})^{\frac{2}{3}} = (1)^{\frac{2}{3}}$

$A = \int_0^1 (\sqrt{x} - x^2) = \frac{2x^{\frac{3}{2}}}{3} - \frac{x^3}{3}\Big|_0^1 = 0.3\overline{3}$

$\bar{x} = \frac{1}{0.33}\int_0^1 x(\sqrt{x} - x^2) = \frac{1}{0.33}\int_0^1 x^{\frac{3}{2}} - x^3 = \frac{1}{0.33}\left[\frac{2x^{\frac{5}{2}}}{5} - \frac{x^4}{4}\Big|_0^1\right]$

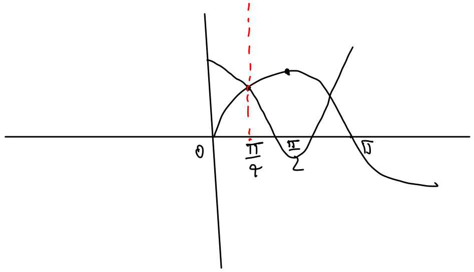
≈ 0.45

$\bar{y} = \frac{1}{0.33}\cdot\frac{1}{2}\int_0^1 [(\sqrt{x})^2 - (x^2)^2 = 1.5\int_0^1 x - x^4 = \frac{1}{6}\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1$

$= \frac{9}{20}$

centroid = (0.45, 0.45)

b. $y = \sin x$, $y = \cos x$, $x = 0$, $x = \frac{\pi}{4}$



Area $= \int_0^{\frac{\pi}{4}} (\cos x) - \sin x$

$= \left[\sin x + \cos x\right]_0^{\frac{\pi}{4}}$

$= -1 + \sqrt{2}$

$$= -1+\sqrt{2}$$

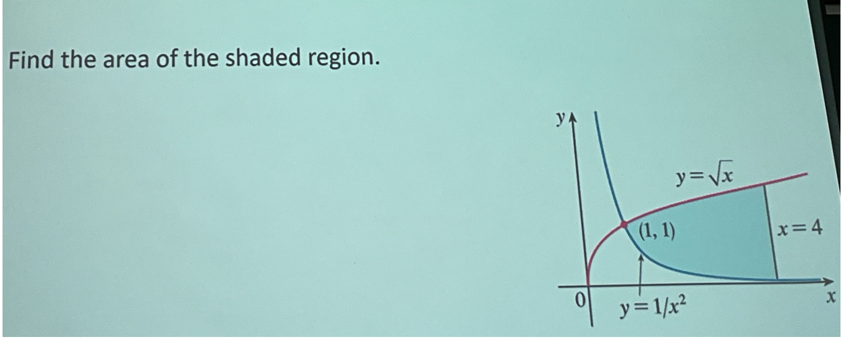
$$\bar{x} = \frac{1}{-1+\sqrt{2}} \int_{0}^{\frac{\pi}{4}} x\left[\cos x - \sin x\right]$$

$$\approx 0.267$$

$$\bar{y} = \frac{1}{-1+\sqrt{2}} \cdot \frac{1}{2} \int_{0}^{\frac{\pi}{4}} \underbrace{(\cos(x))^2}_{\text{กำลัง 2 แยกกัน}} - \underbrace{(\sin x)^2}_{\text{แยก}}$$

$$\approx 0.6$$

$$\boxed{\text{Centroid} = (0.267, 0.6}$$



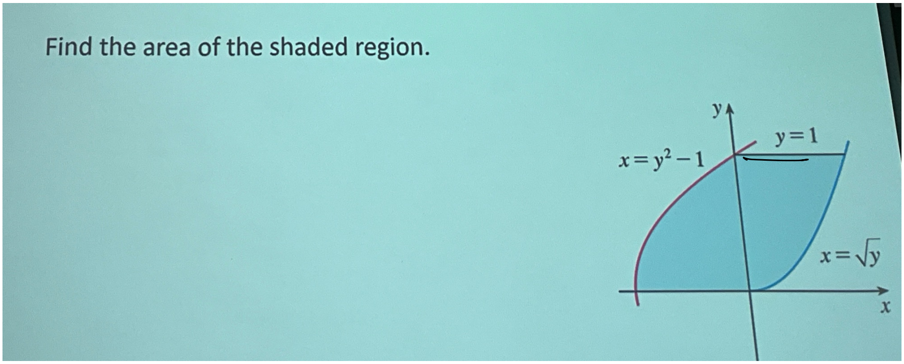
Find the area of the shaded region.

$\frac{1}{x^2} = \sqrt{x}$

$1 = x^{\frac{5}{2}}$

$x = 1$

: $\int_1^4 \left(\sqrt{x} - \frac{1}{x^2}\right) dx = \frac{2}{3} x^{\frac{3}{2}} + x^{-1} \Big|_1^4 = \frac{47}{12}$



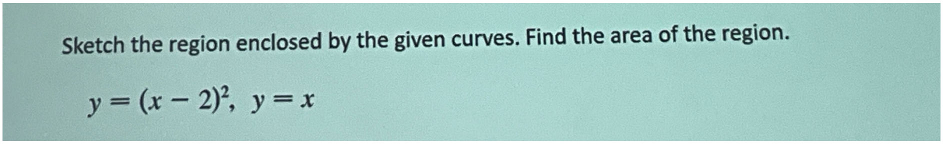
Find the area of the shaded region.

$= \int_0^1 (x_R - x_L)\, dx$

$= \int_0^1 \sqrt{y} - (y^2 - 1)\, dy$

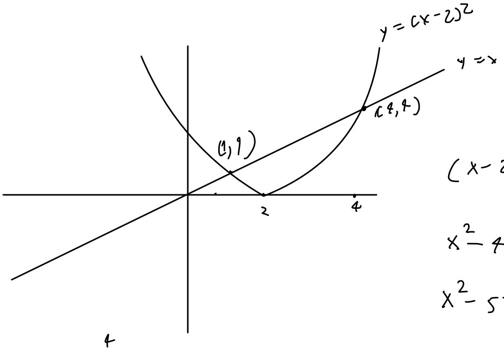
$= \int_0^1 \sqrt{y} - y^2 + 1 \quad dy$

$= \frac{2y^{\frac{3}{2}}}{3} - \frac{y^3}{3} + y \Big]_0^1 = 1.3\overline{3}$



Sketch the region enclosed by the given curves. Find the area of the region.

$y = (x-2)^2, \quad y = x$



$(x-2)^2 = x$

$x^2 - 4x + 4 = x$

$x^2 - 5x + 4 = 0$

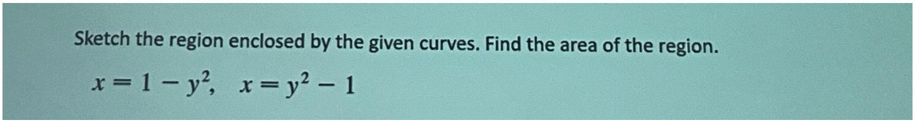
$(x-4)(x-1) = 0$

$x = 4, 1$

$$\int_1^4 (x - (x-2)^2)\,dx$$

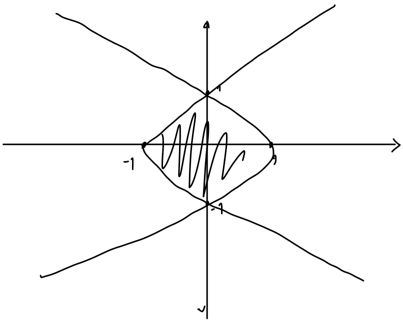
$$= \frac{9}{2}$$

*



Sketch the region enclosed by the given curves. Find the area of the region.

$x = 1 - y^2, \quad x = y^2 - 1$



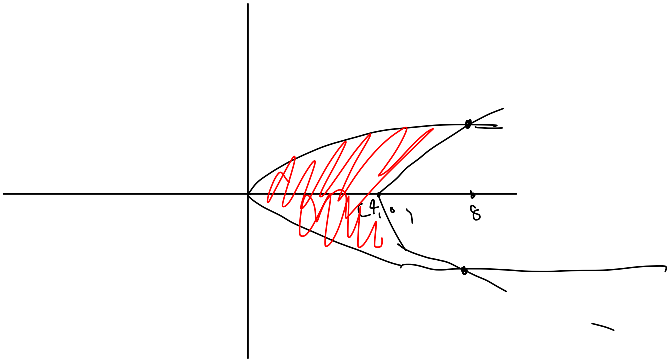
$(0,0)$

$$= \int_{-1}^{1} [x_R - x_L]\,dy \rightarrow A = \int_{-1}^{1} [(1-y^2) - (y^2-1)]\,dy$$

$$= \int_{-1}^{1} 2(1-y^2)\,dy = 4\int_0^1 (1-y^2)\,dy = \frac{8}{3}$$

Sketch the region enclosed by the given curves. Find the area of the region.

$x = 2y^2, \quad x = 4 + y^2$



$x = 4+y^2$

$x = 2y^2$

$y^2 = x-8$

$y = \sqrt{x-8}$

4

$2y^2 = x$

$y^2 = \frac{x}{2}$

$y = \sqrt{\frac{x}{2}}$

$4+y^2 = 2y^2$

$4 = 2y^2 - y^2$

$4 = y^2$

$y = \pm 2$

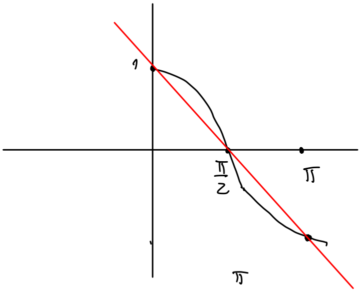
$$A = \int_{-2}^{2} \left[(4+y^2) - 2y^2 \right] dy = 2 \int_{0}^{2} (4-y^2)\, dy = \boxed{\frac{32}{3}}$$

$\downarrow$

$[x_R - x_L]$

Find Area

$y = \cos x$, $y = 1 - \frac{2x}{\pi}$



$1 - \frac{2}{\pi}x = \cos x$

$1 = \cos x + \frac{2}{\pi}x$

$x = \pi, \frac{\pi}{2}, 0 \longrightarrow$ check 0 ก่อน

$$\int_{\frac{\pi}{2}}^{\pi} \left(1 - \frac{2}{\pi}x - \cos x\right) dx$$

$$= x - \frac{x^2}{\pi} - \sin x \Big|_{\frac{\pi}{2}}^{\pi} \approx 0.2146$$

$$= \Big[\underbrace{\pi - \pi - \sin \pi}_{0}\Big] - \left[\frac{\pi}{2} - \frac{\left(\frac{\pi}{2}\right)^2}{\pi} - \sin\left(\frac{\pi}{2}\right)\right]$$

$$- \left[\frac{\pi}{2} - \frac{\pi}{4} - 1\right]$$

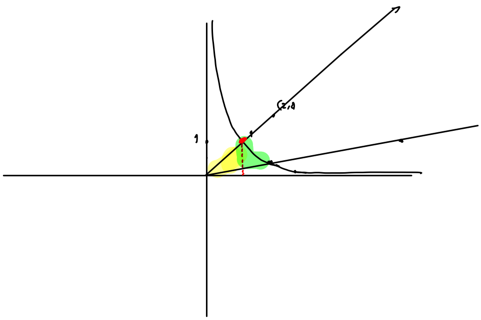
$$= -\left[\frac{\pi}{4} - 1\right] = -\left[\frac{\pi - 4}{4}\right] \approx \frac{-\pi + 4}{4}$$

symmetry to 0 to $\frac{\pi}{2}$

so $2\left[\frac{-\pi + 4}{4}\right] = \frac{-\pi + 4}{2}$

7. Sketch Find area of the region

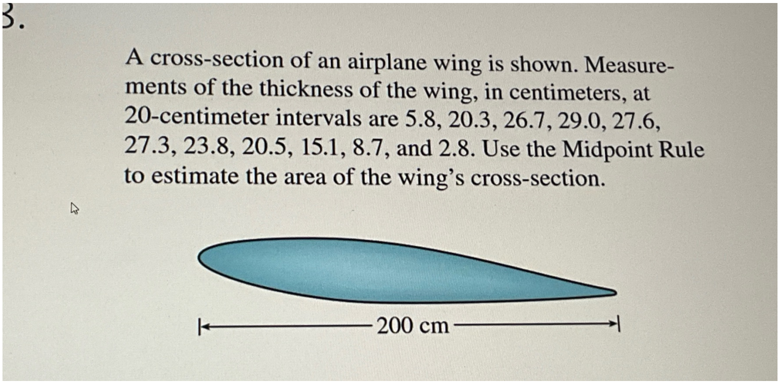
$y = \frac{1}{x^2}$, $y = x$, $y = \frac{1}{8} x$



$$A = \int_0^1 \left(x - \frac{x}{8}\right) dx + \int_1^2 \left(\frac{1}{x^2} - \frac{1}{8}\right) dx$$

$$= \int_0^1 \frac{7x}{8} \, dx + \int_1^2 \left(x^{-2} - \frac{x}{8}\right) dx$$

$$= \left[\frac{7}{16} x^2\right]_0^1 + \left[-\frac{1}{x} - \frac{1}{16} x^2\right]_1^2 = \frac{3}{4}$$



3.

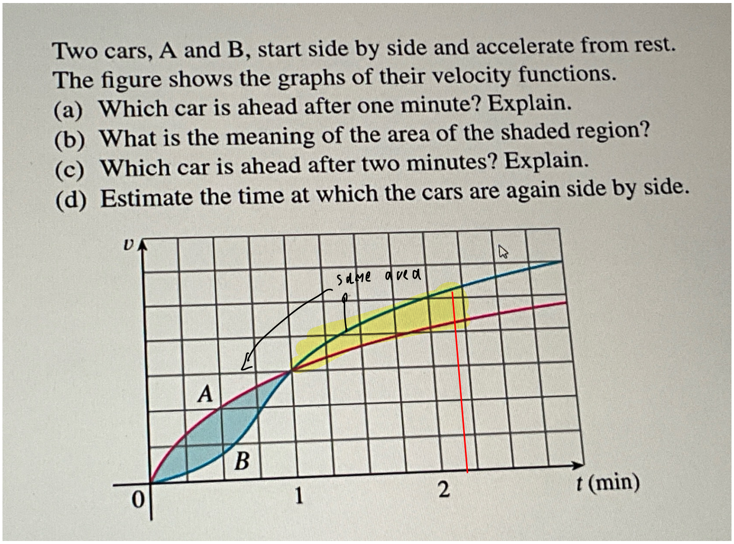
A cross-section of an airplane wing is shown. Measurements of the thickness of the wing, in centimeters, at 20-centimeter intervals are 5.8, 20.3, 26.7, 29.0, 27.6, 27.3, 23.8, 20.5, 15.1, 8.7, and 2.8. Use the Midpoint Rule to estimate the area of the wing's cross-section.

$$\Delta x = \frac{200 - 0}{n} = \frac{200}{5} = 40$$

$n \rightsquigarrow$ let $n = 5$

$$= 40 \left[f(20) + f(60) + f(100) + f(140) + f(180) \right]$$

$$\approx 40(105.8) \approx 4232.$$



Two cars, A and B, start side by side and accelerate from rest. The figure shows the graphs of their velocity functions.

(a) Which car is ahead after one minute? Explain.

(b) What is the meaning of the area of the shaded region?

(c) Which car is ahead after two minutes? Explain.

(d) Estimate the time at which the cars are again side by side.

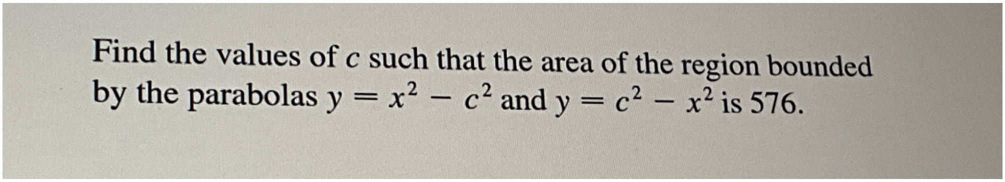
(a) Car A → during the 1 minute interval it have higher velocity.

↓ Area under the curve is bigger.

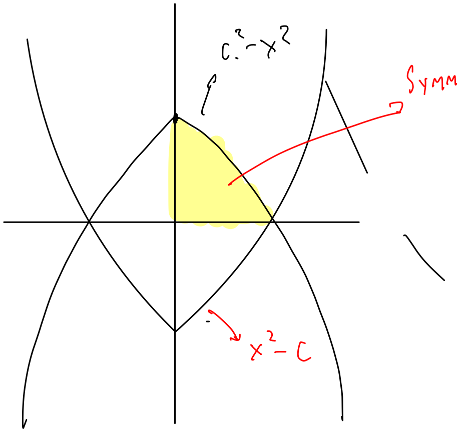
(b) The distance difference between the two car after 1 minute

(c.) after 2 minute car a is still ahead as it have greater area under curve.

(d) around 2.2 minut



Find the values of c such that the area of the region bounded by the parabolas $y = x^2 - c^2$ and $y = c^2 - x^2$ is 576.



Symmetry = so just find 1 Area and Multiply by 4.

$$= 4\int_0^c c^2 - x^2 \, dx$$

$$= 4\left[c^2x - \frac{x^3}{3} \right]_0^c$$

$9 - x^2$

$$= 4\left[c^3 - \frac{c^3}{3} \right] = 4\left[\frac{2c^3}{3} \right]$$

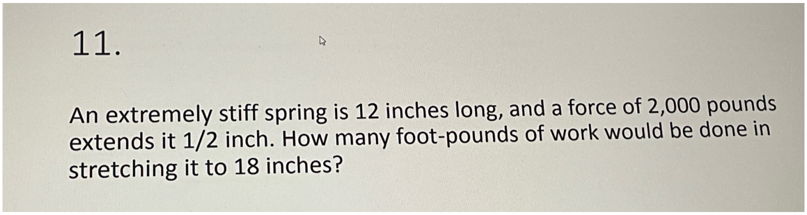
$$= \frac{8c^3}{3}$$

$$\frac{8c^3}{3} = 576$$

$$8c^3 = 1728$$

$$c^3 = 216$$

$$\underline{\underline{c = 6}}$$



11.

An extremely stiff spring is 12 inches long, and a force of 2,000 pounds extends it 1/2 inch. How many foot-pounds of work would be done in stretching it to 18 inches?

$f(x) = kx$

$2000 = k(\frac{1}{2})$

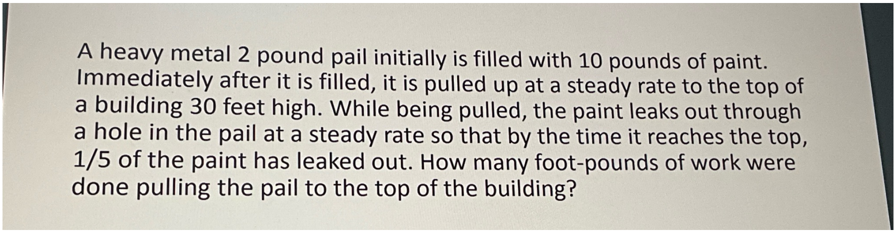
$2000 = \frac{1}{2}k \qquad k = 4000$

$f(x) = 4000x$

ดูแค่ระยะทางทั้งหมด ไม่ต้องดูที่ coordinate.

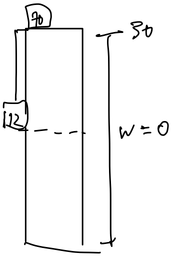
$=$

$= \int_0^6 4000x \, dx = 2000x^2 \Big|_0^6 = 72000$



A heavy metal 2 pound pail initially is filled with 10 pounds of paint. Immediately after it is filled, it is pulled up at a steady rate to the top of a building 30 feet high. While being pulled, the paint leaks out through a hole in the pail at a steady rate so that by the time it reaches the top, 1/5 of the paint has leaked out. How many foot-pounds of work were done pulling the pail to the top of the building?

✱✱



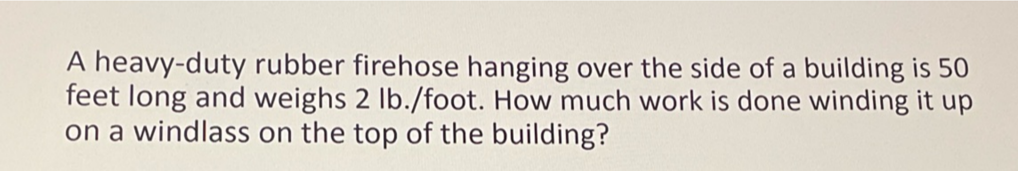
at the end paint = 8 pound

w(h) = weight as height h

$\left.\begin{array}{l} w(0) = 12 \\ w(30) = 10 \end{array}\right\} \quad w(h) = 12 - \frac{h}{15}$

work $\int_0^{30} \left(12 - \frac{h}{15}\right) dh = 12h - \frac{h^2}{30} \Big|_0^{30} = 330$ ft -lbs

weight is in newton and pounds = Force

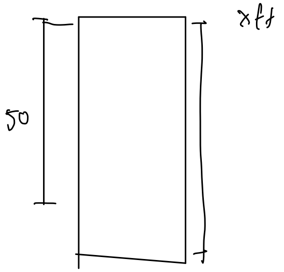


A heavy-duty rubber firehose hanging over the side of a building is 50 feet long and weighs 2 lb./foot. How much work is done winding it up on a windlass on the top of the building?

$F = wd$

$=$

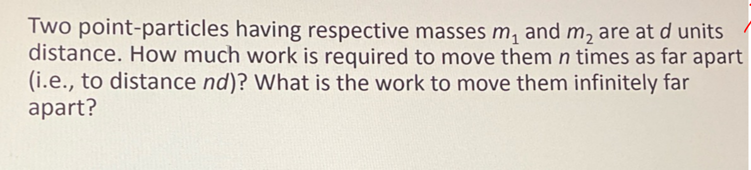
$= 2\,\Delta x$



$= \int_0^{50} 2x\,dx$

$= x^2 \Big]_0^{50}$

$\simeq 2500$



Two point-particles having respective masses m_1 and m_2 are at d units distance. How much work is required to move them n times as far apart (i.e., to distance nd)? What is the work to move them infinitely far apart?

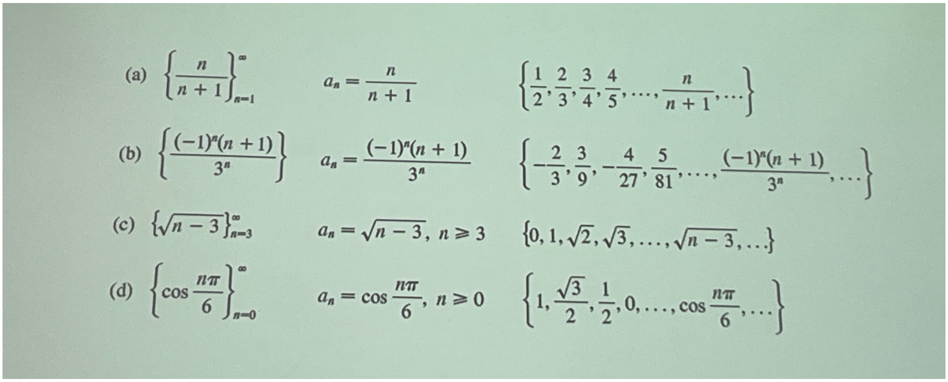
$\boxed{\dfrac{g M_1 M_2}{x^2}} \longrightarrow$ Distance from d to nd

$$\text{Work} = \int_{d}^{nd} \frac{g M_1 M_2}{x^2}\, dx = -\frac{g M_1 M_2}{x}\Bigg|_{d}^{nd} = \frac{g M_1 M_2}{d}\left(\frac{n-1}{n}\right)$$

Sequence

$S_n = a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$

first term Second term

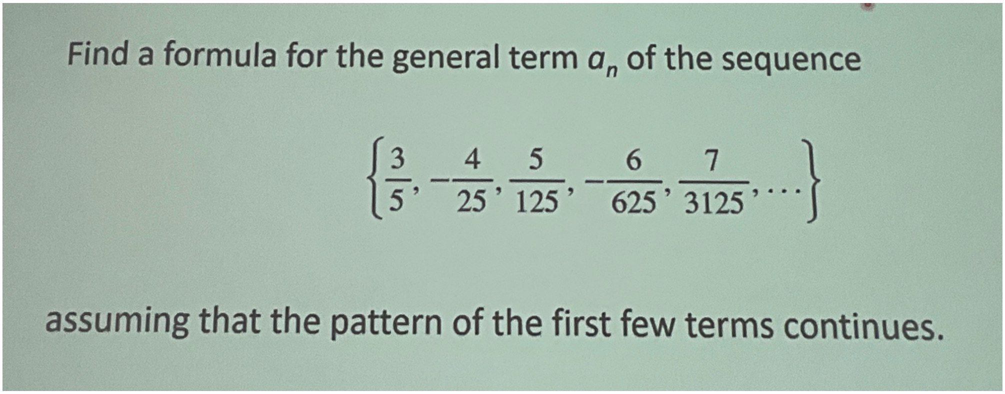


(a) $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ $a_n = \frac{n}{n+1}$ $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots\right\}$

(b) $\left\{\frac{(-1)^n(n+1)}{3^n}\right\}$ $a_n = \frac{(-1)^n(n+1)}{3^n}$ $\left\{-\frac{2}{3}, \frac{3}{9}, -\frac{4}{27}, \frac{5}{81}, \ldots, \frac{(-1)^n(n+1)}{3^n}, \ldots\right\}$

(c) $\left\{\sqrt{n-3}\right\}_{n=3}^{\infty}$ $a_n = \sqrt{n-3},\ n \geq 3$ $\left\{0, 1, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n-3}, \ldots\right\}$

(d) $\left\{\cos\frac{n\pi}{6}\right\}_{n=0}^{\infty}$ $a_n = \cos\frac{n\pi}{6},\ n \geq 0$ $\left\{1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, \ldots, \cos\frac{n\pi}{6}, \ldots\right\}$

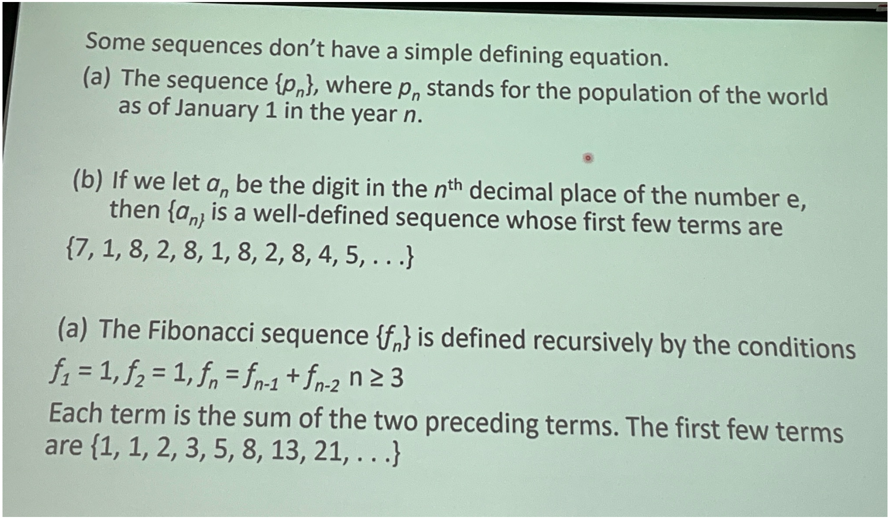


Find a formula for the general term a_n of the sequence

$$\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3125}, \ldots\right\}$$

assuming that the pattern of the first few terms continues.

$$a_n = (-1)^{n-1}\left(\frac{n+2}{5^n}\right)$$



Some sequences don't have a simple defining equation.

(a) The sequence $\{p_n\}$, where p_n stands for the population of the world as of January 1 in the year n.

(b) If we let a_n be the digit in the n^{th} decimal place of the number e, then $\{a_n\}$ is a well-defined sequence whose first few terms are

$\{7, 1, 8, 2, 8, 1, 8, 2, 8, 4, 5, . . .\}$

(a) The Fibonacci sequence $\{f_n\}$ is defined recursively by the conditions

$f_1 = 1, f_2 = 1, f_n = f_{n-1} + f_{n-2}$ n ≥ 3

Each term is the sum of the two preceding terms. The first few terms are $\{1, 1, 2, 3, 5, 8, 13, 21, . . .\}$

$$\lim_{n \to \infty} \frac{n}{n+1} = 1$$

A sequence $\{a_n\}$ has the limit L and we write

$$\lim_{n \to \infty} a_n = \underset{\text{Limit}}{\underset{\downarrow}{L}} \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

If limit exist ⟶ S_n converge

↓

doesnot exist ⟶ S_n diverge.

Limit of a sequence =

$\lim\limits_{n\to\infty} a_n = L$ or $a_n \to L$ an $n \to \infty$

For every $\varepsilon > 0$ there is a corresponding integer N

If $n > N$ then $|a_n - L| < \varepsilon$

Theorem 1.

$n =$ Integer

If $\lim\limits_{x\to\infty} f(x) = L$ and $f(n) = a_n \to$ then $\lim\limits_{n\to\infty} a_n = L$

Ex. Find $\lim\limits_{n\to\infty} \frac{1}{n^r}$ (If $r > 0$]

$= 0$

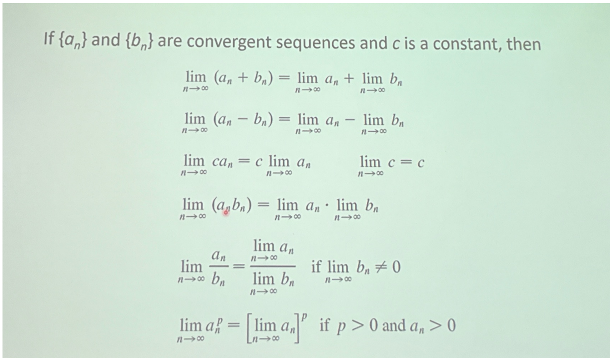
\#

From the theorem we know that

$\lim\limits_{n\to\infty} \frac{1}{x^r} = 0$ when $\to$ $r > 0$

Definition

$\lim\limits_{n\to\infty} a_n = \infty$ mean that for every positive number M therere is integer N that if $n > N$ then $a_n > M$.



If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n \qquad \lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n} \quad \text{if } \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \quad \text{if } p > 0 \text{ and } a_n > 0$$

Squeeze theorem

If $a_n \le b_n \le c_n$ for $n \ge n_0$

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$.

Theorem

If $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$

Ex.

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}}$$

$$= \frac{1}{1+0} = 1 \quad \#$$

Ex2. Is the sequence converge or diverge

$$a_n = \frac{n}{\sqrt{10+n}} \qquad \lim_{n\to\infty} a_n = L$$

$$\lim_{n\to\infty} \frac{n}{\sqrt{10+n}} = \frac{\frac{n}{n}}{\frac{\sqrt{10+n}}{n}} = \frac{1}{\sqrt{\frac{10}{n^2}+\frac{n}{n^2}}} = \infty$$

diverge.

Find

$$\lim_{n \to \infty} \frac{\ln n}{n} = \frac{\infty}{\infty} = \frac{1}{n}$$

↓

$$\lim_{n \to \infty} \frac{1}{n} = 0$$ converge.

Find Determine whether this sequence converge or diverge

$$a_n = (-1)^n$$!!! don't mess up

↓

Oscillate beween 1, -1 so Diverge

$$\lim_{n \to \infty} \frac{(-1)^n}{n} .$$

$$= \lim_{n \to \infty} \left| \frac{(-1)^n}{n} \right| = \lim_{n \to \infty} \frac{1}{n} = 0$$

↓

converge

Theorem

If $\lim\limits_{n \to \infty} a_n = L$ and the function f is continuous at L, then

$$\lim_{n \to \infty} f(a_n) = f(L)$$

Find $\lim_{n \to \infty} \sin\left(\frac{\pi}{n}\right)$ ↗ sin, continuous at 0.

$$= \sin\left[\lim_{n \to \infty} \frac{\pi}{n}\right] = \sin(0) = 0$$

#

Discuss convergence $a_n = \frac{n!}{n^n}$

↓

$$= \frac{n \cdot (n-1) \cdot (n-2)(n-3) \cdot \cdots 3 \times 2 \times 1}{n^n}$$

$= ex: n=3 \quad \frac{3 \times 2 \times 1}{3 \times 3 3}$

$5: \quad \frac{5 \times 4 \times 3 \times 2 \times 1}{5 \times 5 \times 5 \times 5 5}$

$$\frac{n!}{n^n} \leq \frac{1}{n}$$

$$0 \leq a_n \leq \frac{1}{n}$$

$\lim_{n \to \infty} \frac{1}{n} = 0 \longrightarrow$ Thus by the squeeze theorem, this sequence converge to zero.

Find r for convergent $\{r^n\}$

Theorem $\lim_{n\to\infty} r^n = \begin{cases} \infty & \text{if } r > 1 \\ 0 & \text{if } 0 < r < 1 \end{cases}$

If $-1 < r < 0$, then $0 < |n| < 1$, so

$$\lim_{n\to\infty} |r^n| = \lim_{n\to\infty} |r|^n = 0 \xrightarrow{\text{Theorem}} \lim_{n\to\infty} r^n = 0$$

If $r \le -1$, then $\{r^n\}$ diverges

so $-1 < r \le 1$

For Future use the sequence

$r^n \to$ converge if $-1 < r \le 1$

and diverge for all other value of r

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

Increasing, Decreasing, and Monotic sequence

$a_n \to$ increasing if $a_n < a_{n+1}$ for $n \ge 1$,

that $a_1 < a_2 < a_3 < \dots$

Note

: It is monotonic if it is either increasing or decreasing.

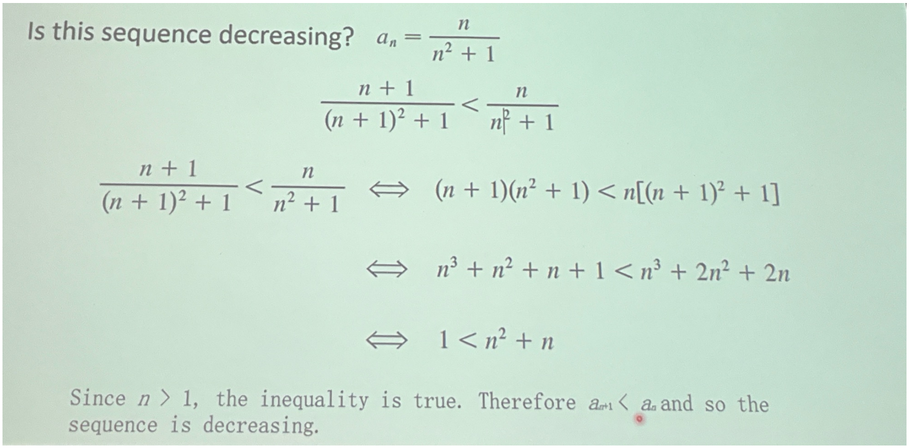
decreasing if $a_n > a_{n+1} \to n \ge 1$

Is this sequence decrease $a_n = \dfrac{n}{n^2+1}$

$\Downarrow$

$$\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1}$$

$\Downarrow$



Is this sequence decreasing? $a_n = \dfrac{n}{n^2+1}$

$$\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1}$$

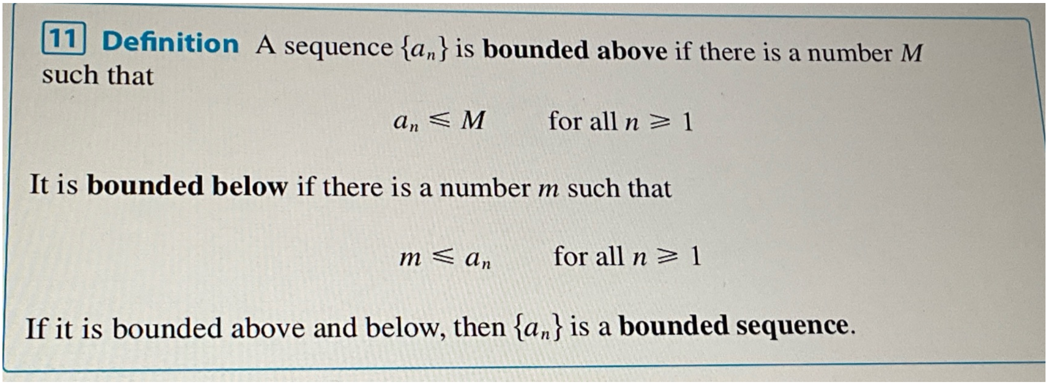
$$\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1} \iff (n+1)(n^2+1) < n[(n+1)^2+1]$$

$$\iff n^3+n^2+n+1 < n^3+2n^2+2n$$

$$\iff 1 < n^2+n$$

Since $n > 1$, the inequality is true. Therefore $a_{n+1} < a_n$ and so the sequence is decreasing.

Bounded Sequence



11 Definition A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

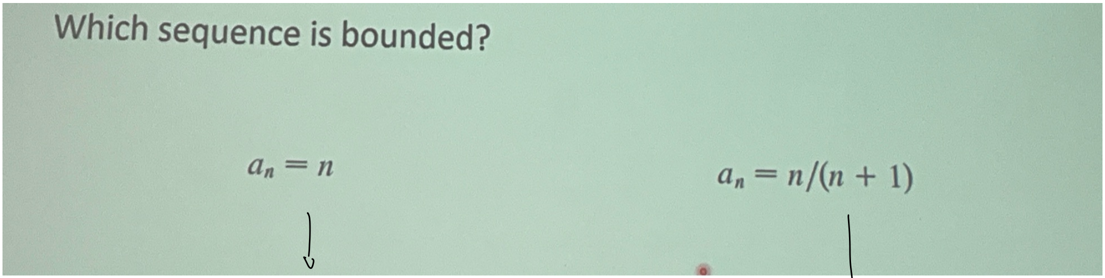
$$a_n \leqslant M \qquad \text{for all } n \geqslant 1$$

It is **bounded below** if there is a number m such that

$$m \leqslant a_n \qquad \text{for all } n \geqslant 1$$

If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.

Every bounded, monotonic sequence is convergent.



Which sequence is bounded?

$a_n = n$ $\qquad\qquad$ $a_n = n/(n+1)$

$\downarrow$ $\qquad\qquad$ $\downarrow$

Bounded Below $\qquad\qquad$ Bounded

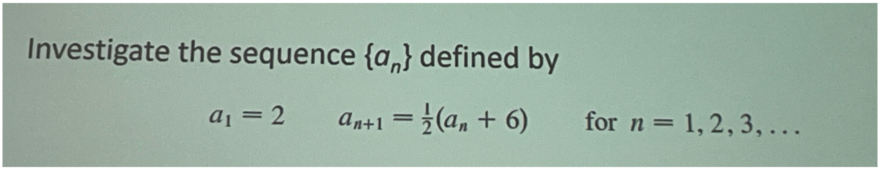
$(a_n > 0)$

Induction

1. S_1 is true

2. S_{k+1} is true whenever S_k is true.

Then S_n is true for all positive integers



Investigate the sequence $\{a_n\}$ defined by

$$a_1 = 2 \qquad a_{n+1} = \frac{1}{2}(a_n + 6) \qquad \text{for } n = 1, 2, 3, \ldots$$

$a_1 = 2$

$a_2 = \frac{1}{2}(a_1 + 6)$

2

$= \frac{1}{2}(2+6) = 4$

1

$a_3 = \frac{1}{2}(a_2 + 6) = 5$

0.5

$a_4 = \frac{1}{2}(5+6) = \frac{11}{2} = 5.5$

.25

$a_5 = \frac{1}{\cancel{2}}(5.5+\cancel{6}) = \frac{1}{\cancel{2}}(\overset{5}{\cancel{11.5}}) = 5.75$

0.125

$a_6 = 5.875$

0.0625

$a_7 = 5.9375$

Might converge to 6.

To prove the sequence is increase we can use induction.

that $a_n < a_{n+1}$

Test for n=1 $a_{1+1} = \frac{1}{2}(2+6) = 4$

$a_1 < 4 \longrightarrow a_1 = 2$

Prove #

Now assume n is true such that n = n+1 is also true.

If n = n+1

then

$a_{n+1} < a_{n+2}$ ⟶ try to prove.

start $a_n < a_{n+1}$

$$\frac{1}{2}(a_n + 6) < (a_{n+1} + 6)\frac{1}{2}$$

$$= \frac{1}{2}(a_n + 6) < (a_{n+1} + 6)\frac{1}{2}$$

<u>According to Formula</u> $a_{n+2} = \frac{1}{2}(a_{n+1} + 6)$, $a_{n+1} = \frac{1}{2}(a_n + 6)$

so $a_{n+1} < a_{n+2}$

∴ Prove increase

Check by induction if the sequence is bounded by 6.

$$a_K < 6$$

test $k = 1 \longrightarrow a_1 < 6$ -> true

assume k is true such that $a_{K+1} < 6$

$$a_K < 6$$

$$a_K + 6 < 12$$

$$\frac{1}{2}(a_K + 6) < \frac{1}{2}(12)$$

$$a_{K+1} < 6 \;\Big\}$$ Prove

Check the limit to see which number it converge too.

$$\lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \frac{1}{2}(a_n + 6) = \frac{1}{2} \lim_{n \to \infty} a_n + 6 = \frac{1}{2}(L + 6)$$

Since $a_n \to L$, it also show that $a_{n+1} \to L$ (as $n \to \infty$, $n+1 \to \infty$)

$$L = \frac{1}{2}(L + 6)$$

$$\downarrow$$

$$\underline{\underline{L = 6}}$$

Series

If we add the terms of an infinite sequence.

Infinite Series = $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$

Partial Sums → $s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$

Definition : $\sum_{n=1}^{\infty} a_n \longrightarrow \underline{s_n} = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n .$

↓

If the sequence is convergent and $\lim_{n \to \infty} s_n = s$ → real number

then the series $\sum a_n$ is called convergent

↓

so $\sum_{n=1}^{\infty} a_n = s$ → converge

Note!

$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} \sum_{i=1}^{n} a_i$

If it diverge (อรัญ) → thus series also diverge.

Ex. $s_n = \frac{2n}{3n+5}$ → Note: suppose we know that the sum of the first n terms of the series $\sum_{n=1}^{\infty} a_n$

↓

Sum of series is $\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{2n}{3n+5} = \frac{2}{3+\frac{5}{2}} \approx \frac{2}{3}$

Geometric Series

$\sum_{n=1}^{\infty} ar^{n-1}$ $a \neq 0$

r = common ratio

$$S_n = \frac{a(1-r^n)}{1-r} \rightarrow \text{If } -1 < r < 1$$

$\lim_{n\to\infty} r^n = \frac{a}{1-r} \longrightarrow$!!! when $|r| < 1$ $\longrightarrow$ series converge and sum

Sum $\boxed{= \frac{a}{1-r}} \longrightarrow \lim_{n\to\infty} r^n$

When $r \leq -1$ or $r > 1$. the sequence $\{r^n\}$ is divergent.

Example Find sum of

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$$

first term $= 5$ $r = -\frac{2}{3}$ as $\left|-\frac{2}{3}\right| < 1$ then the

→ between $-1 < a_n < 1$

after $a_n \leq -1$ → diverge

Series converge and sum $= \frac{5}{1-(-\frac{2}{3})} = \frac{5}{\frac{5}{3}} = 3$ #

*

Ex. 2 is the series $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$ converge or diverge

as $\sum_{n=1}^{\infty} ar^{n-1}$ $a \neq 0$

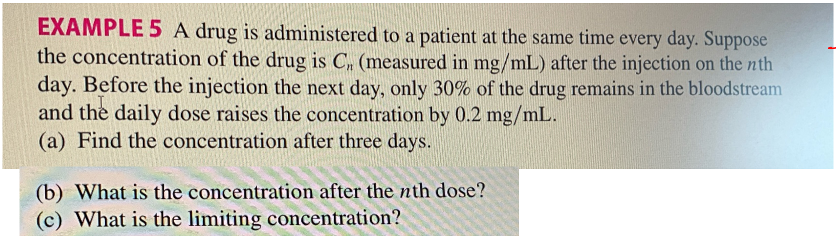
$\downarrow$

$2^{2n} 3^{1-n} \to$ make simiral

$= 2^{2(n)} 3^{-(n-1)}$

$= \sum_{n=1}^{\infty} \frac{4^n}{3^{n-1}} \to \sum_{n=1}^{\infty} \frac{4^1 \times 4^{n-1}}{3^{n-1}} = 4\left(\frac{4}{3}\right)^{n-1}$

$a = 4$, $r = \frac{4}{3} \to \left|\frac{4}{3}\right| > 1 \to$ Diverge



EXAMPLE 5 A drug is administered to a patient at the same time every day. Suppose the concentration of the drug is C_n (measured in mg/mL) after the injection on the nth day. Before the injection the next day, only 30% of the drug remains in the bloodstream and the daily dose raises the concentration by 0.2 mg/mL.

(a) Find the concentration after three days.

(b) What is the concentration after the nth dose?

(c) What is the limiting concentration?

(a) = Concentration = C_n

Daily dose

Before inject next day = $0.3C_n + 0.2$

so $C_{n+1} = 0.3C_n + 0.2$

Three days : $C_0 = C_1 = 0.3C_0 + 0.2 = 0.2$

$C_n = C_1$

$C_1 + 1 = C_2 = 0.3(0.2) + (0.2) = 0.26$

$C_n = C_2$

$C_2 + 1 = C_3 = 0.3(0.26) + 0.2$

(b) After the nth dose

$$C_n = 0.2 + 0.2(0.3) + 0.2(0.3)^2 + \dots + 0.2(0.3)^{n-1}$$

as $a = 0.2$, $r = 0.3$ $\quad S_n = \dfrac{a(1-r^n)}{1-r}$ (finite geometric series)

$$C_n = \frac{0.2[1-(0.3)^n]}{1-0.3} = \frac{2}{7}[1-(0.3)^n]$$

(c) limit of concentration $\qquad (0.3)^{\infty} = 0 \quad .3 < 0 < 1 .$

$$\lim_{n\to\infty} C_n = \lim_{n\to\infty} \frac{2}{7}[1-\overset{\nearrow 0}{(0.3)^n}] = \frac{2}{7} \text{ mg / mL}$$

Write the number $2.3\overline{17} = 2.317\,17\,17$ as a ratio of integers. $**$

$$2.3171717 = 2.3 + \frac{17}{10^3} + \frac{17}{10^5} + \frac{17}{10^7} + \dots$$

$$a = \frac{17}{10^3} \qquad r = \frac{1}{10^2}$$

as $|r| < 1$ so it convergent

$$\therefore 2.3\overline{17} = 2.3 + \frac{\frac{17}{10^3}}{1-\frac{1}{10^2}} \overset{\frac{a}{1-r}}{=} 2.3 + \frac{\frac{17}{1000}}{\frac{99}{100}}$$

$$= \frac{23}{10} + \frac{17}{990} = \frac{1147}{495}$$

Find the sum of Series $\sum_{n=0}^{\infty} x^n$, where $|x|<1$

$n = 0$

$= x^0 + x^1 + x^2 + x^3 + \dots$

$= 1 + x + x^2 + x^3 + \dots$

$a = 1$, $r = x$ as $|x| < 1$

$\downarrow$

converge

$$\text{sum} = \frac{1}{1-x}$$

Ex. 8. Show that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and find its sum

$\downarrow \quad \downarrow$

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$= \sum_{i=1}^{n} \frac{1}{i} - \frac{1}{i+1}$$

$$= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}\right)$$

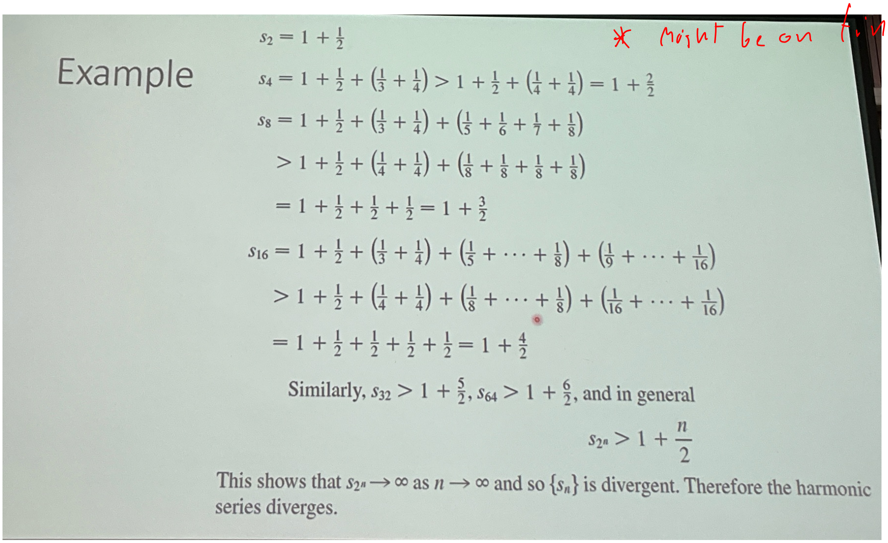
$$+ \dots \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

$$= \lim_{n \to \infty} 1 - \frac{1}{n+1} = 1 - 0 = 1$$ → Thus converge.

6 Theorem If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Show $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\cdot$ → Harmonic Series

* Might be on final



Example

$$s_2 = 1 + \tfrac{1}{2}$$

$$s_4 = 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) > 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) = 1 + \tfrac{2}{2}$$

$$\begin{aligned} s_8 &= 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}\right) \\ &> 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \left(\tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}\right) \\ &= 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = 1 + \tfrac{3}{2} \end{aligned}$$

$$\begin{aligned} s_{16} &= 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \cdots + \tfrac{1}{8}\right) + \left(\tfrac{1}{9} + \cdots + \tfrac{1}{16}\right) \\ &> 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \left(\tfrac{1}{8} + \cdots + \tfrac{1}{8}\right) + \left(\tfrac{1}{16} + \cdots + \tfrac{1}{16}\right) \\ &= 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = 1 + \tfrac{4}{2} \end{aligned}$$

Similarly, $s_{32} > 1 + \frac{5}{2}$, $s_{64} > 1 + \frac{6}{2}$, and in general

$$s_{2^n} > 1 + \frac{n}{2}$$

This shows that $s_{2^n} \to \infty$ as $n \to \infty$ and so $\{s_n\}$ is divergent. Therefore the harmonic series diverges.

7 Test for Divergence If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

!!! → If $\lim_{n\to\infty} a_n = 0$ → then we cannot conclude yet if it converge or diverge.

might diverge.

8 Theorem If $\Sigma\, a_n$ and $\Sigma\, b_n$ are convergent series, then so are the series $\Sigma\, ca_n$ (where c is a constant), $\Sigma\,(a_n + b_n)$, and $\Sigma\,(a_n - b_n)$, and

(i) $\sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n$

(ii) $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

These properties of convergent series follow from the corresponding Limit Laws for

Divergence test

If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$

However the theorem is not true in general. If $\lim_{n\to\infty} a_n = 0$ we cannot conclude that $\sum a_n$ is convergent.

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent

In general If we find that $\lim_{n\to\infty} a_n \neq 0$, we Know that $\sum a_n$ is divergent.

But, if we find that $\lim_{n\to\infty} a_n = 0$, we Know nothing about the convergence or divergence of $\sum a_n$.

Show that the series is divergent. $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$

$$\lim_{n\to\infty} \frac{1}{5+\frac{4}{n^2}} = \frac{1}{5} \longrightarrow \text{Diverge by divergence test}$$

Find Sum of Series $\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right)$

$$= 3\sum_{n=1}^{\infty}\frac{1}{n(n+1)} + \sum_{n=1}^{\infty}\frac{1}{2^n} \longrightarrow \frac{1}{2}\left(\frac{1}{2}\right)^{n-1}$$

$\lim_{n\to\infty} = 0$

Cannot conclude diverge or converge

$\frac{1}{n(n+1)} \to 1$

$a = \frac{1}{2}$ $r = \left|\frac{1}{2}\right| \to < 1$

converge

$$= \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

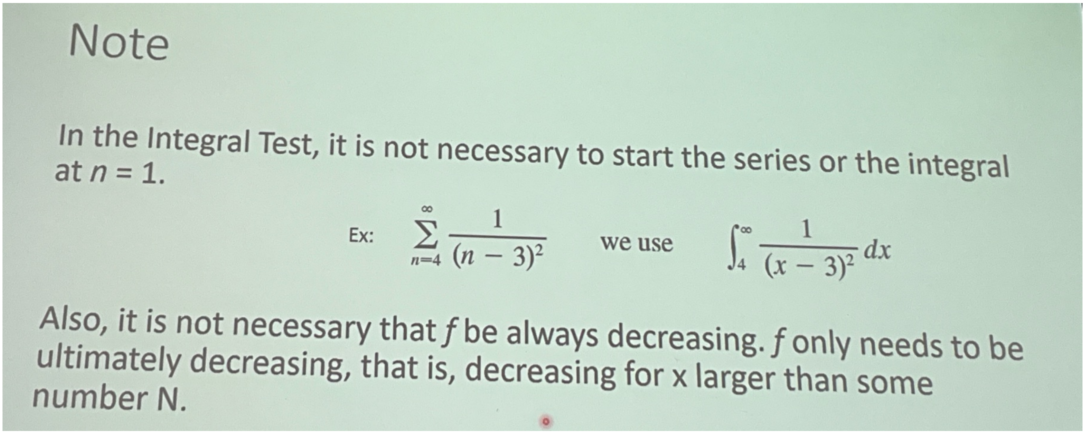
Ans $= 3\cdot 1 + 1 = 4$

Integral test

suppose f is a continuous, positive, decreasing function on $[1, \infty)$

1. If $\int_1^\infty f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

2. If $\int_1^\infty f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent



Note

In the Integral Test, it is not necessary to start the series or the integral at $n = 1$.

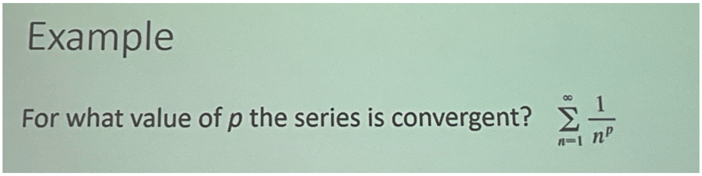
Ex: $\sum_{n=4}^{\infty} \frac{1}{(n-3)^2}$ we use $\int_4^\infty \frac{1}{(x-3)^2}\,dx$

Also, it is not necessary that f be always decreasing. f only needs to be ultimately decreasing, that is, decreasing for x larger than some number N.

Convergent/divergent ? $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

$$= \int_1^\infty \frac{1}{n^2+1} = \lim_{t \to \infty} \arctan^{-1}(n) \Big]_1^t$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$ converge



Example

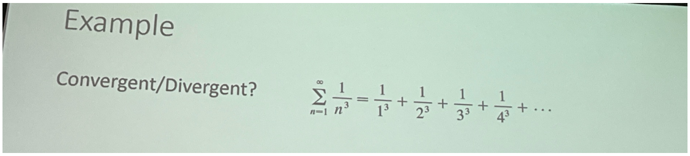
For what value of *p* the series is convergent? $\sum_{n=1}^{\infty} \frac{1}{n^p}$

$\downarrow$

$p > 1$

is convergent if $p>1$ and diverge if $p \leq 1$

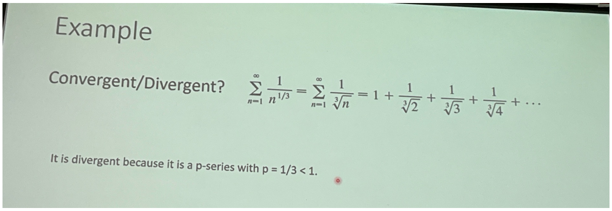
$0 < p \leq 1$



Example

Convergent/Divergent? $\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \dots$

Converge



Example

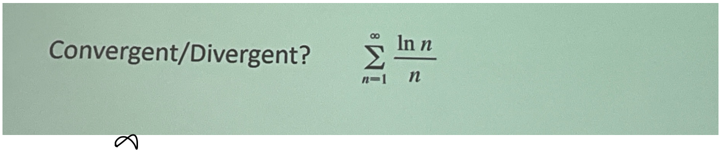
Convergent/Divergent? $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = 1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \dots$

It is divergent because it is a p-series with p = 1/3 < 1.

Diverge

!!) Sum of series is not equal to the value of the integral.

In general: $\sum_{n=1}^{\infty} a_n \neq \int_1^{\infty} f(x)\, dx$



Convergent/Divergent? $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

$$\int_1^\infty \frac{\ln x}{x}\, dx =$$

$v = \ln x \qquad du = \frac{1}{x}\, dx$

$dv = \frac{1}{x} \qquad v = \ln(x)$

$$\int \frac{\ln x}{x} = \ln(x)^2 - \int \ln x \cdot \frac{1}{x}$$

$$I = \ln(x)^2 - I$$

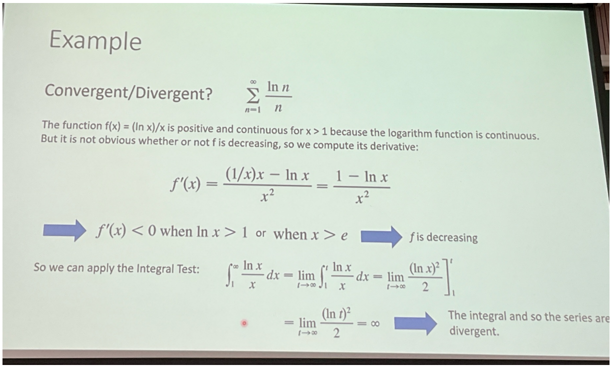
$$= \frac{\ln(x)^2}{2} \Big|_1^\infty$$

$$\lim_{t\to\infty} \frac{\ln(x)^2}{2} \Big|_1^t$$

$$= \frac{\ln(t)^2}{2} - \frac{\ln(1)^2}{2}$$

$= \infty$ diverge.

Other ways.



Example

Convergent/Divergent? $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

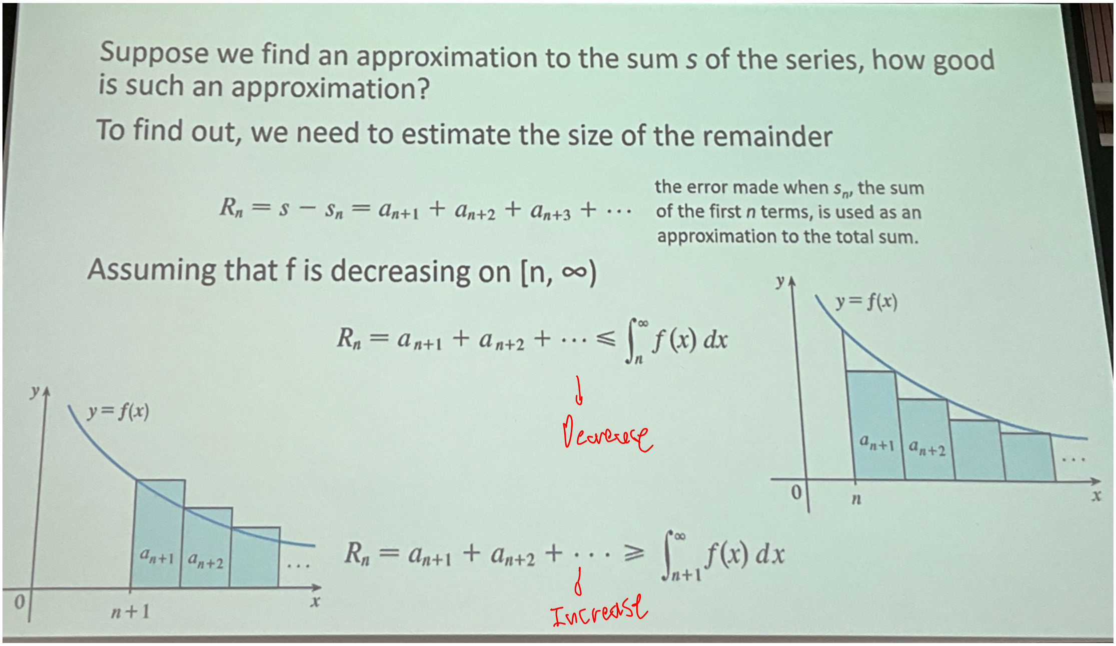
The function f(x) = (ln x)/x is positive and continuous for x > 1 because the logarithm function is continuous. But it is not obvious whether or not f is decreasing, so we compute its derivative:

$$f'(x) = \frac{(1/x)x - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

➡ $f'(x) < 0$ when $\ln x > 1$ or when $x > e$ ➡ f is decreasing

So we can apply the Integral Test: $\int_1^\infty \frac{\ln x}{x}\, dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x}\, dx = \lim_{t\to\infty} \frac{(\ln x)^2}{2}\Big]_1^t$

$= \lim_{t\to\infty} \frac{(\ln t)^2}{2} = \infty$ ➡ The integral and so the series are divergent.



Suppose we find an approximation to the sum s of the series, how good is such an approximation?

To find out, we need to estimate the size of the remainder

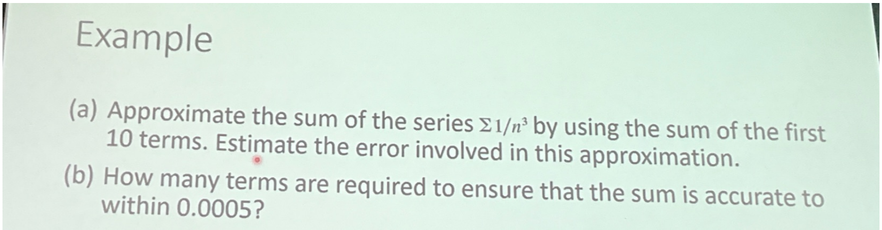
$$R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

the error made when s_n, the sum of the first n terms, is used as an approximation to the total sum.

Assuming that f is decreasing on [n, ∞)

$$R_n = a_{n+1} + a_{n+2} + \underset{\substack{\downarrow \\ \text{Decrease}}}{\cdots} \leqslant \int_n^\infty f(x)\,dx$$

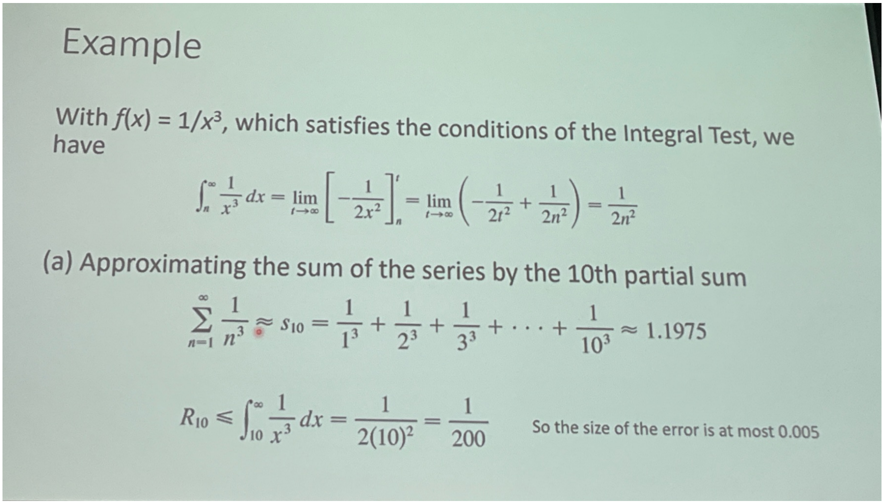
$$R_n = a_{n+1} + a_{n+2} + \underset{\substack{\downarrow \\ \text{Increase}}}{\cdots} \geqslant \int_{n+1}^\infty f(x)\,dx$$



Example

(a) Approximate the sum of the series $\Sigma 1/n^3$ by using the sum of the first 10 terms. Estimate the error involved in this approximation.

(b) How many terms are required to ensure that the sum is accurate to within 0.0005?



Example

With $f(x) = 1/x^3$, which satisfies the conditions of the Integral Test, we have

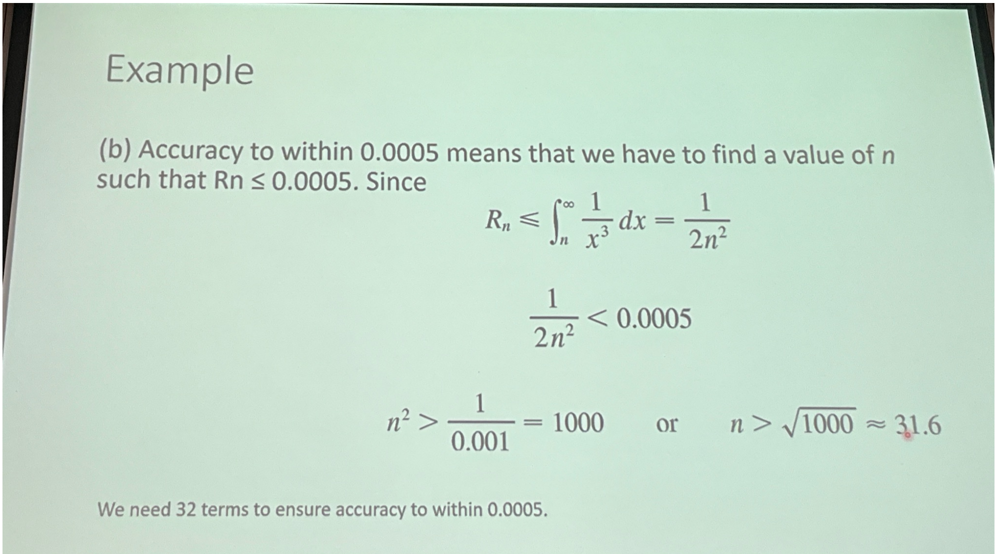
$$\int_n^\infty \frac{1}{x^3}\,dx = \lim_{t\to\infty}\left[-\frac{1}{2x^2}\right]_n^t = \lim_{t\to\infty}\left(-\frac{1}{2t^2} + \frac{1}{2n^2}\right) = \frac{1}{2n^2}$$

(a) Approximating the sum of the series by the 10th partial sum

$$\sum_{n=1}^\infty \frac{1}{n^3} \approx s_{10} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{10^3} \approx 1.1975$$

$$R_{10} \leqslant \int_{10}^\infty \frac{1}{x^3}\,dx = \frac{1}{2(10)^2} = \frac{1}{200}$$

So the size of the error is at most 0.005



Example

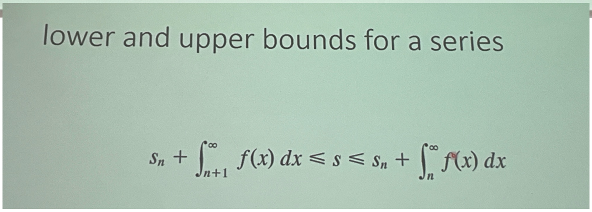
(b) Accuracy to within 0.0005 means that we have to find a value of *n* such that Rn ≤ 0.0005. Since

$$R_n \leqslant \int_n^\infty \frac{1}{x^3}\,dx = \frac{1}{2n^2}$$

$$\frac{1}{2n^2} < 0.0005$$

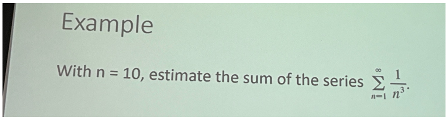
$$n^2 > \frac{1}{0.001} = 1000 \qquad \text{or} \qquad n > \sqrt{1000} \approx 31.6$$

We need 32 terms to ensure accuracy to within 0.0005.



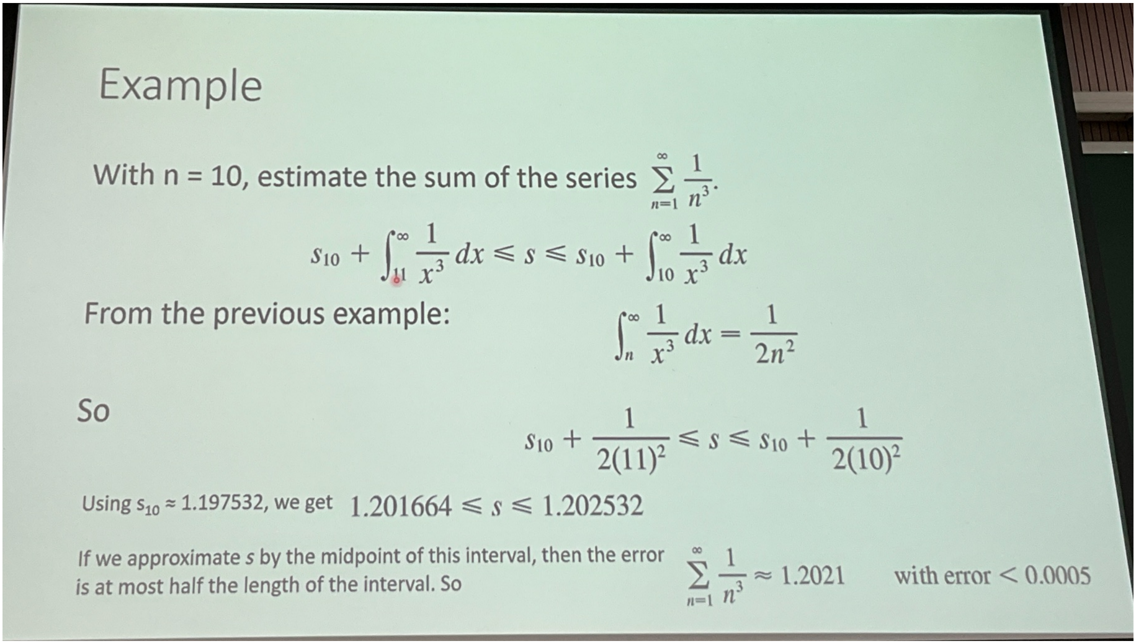
lower and upper bounds for a series

$$s_n + \int_{n+1}^\infty f(x)\,dx \leqslant s \leqslant s_n + \int_n^\infty f(x)\,dx$$



Example

With n = 10, estimate the sum of the series $\sum_{n=1}^\infty \frac{1}{n^3}$.



Example

With n = 10, estimate the sum of the series $\sum_{n=1}^\infty \frac{1}{n^3}$.

$$s_{10} + \int_{11}^\infty \frac{1}{x^3}\,dx \leqslant s \leqslant s_{10} + \int_{10}^\infty \frac{1}{x^3}\,dx$$

From the previous example:

$$\int_n^\infty \frac{1}{x^3}\,dx = \frac{1}{2n^2}$$

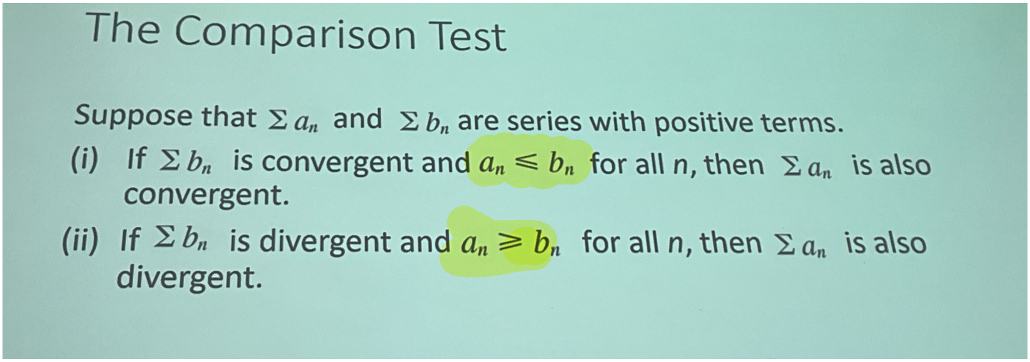
So

$$s_{10} + \frac{1}{2(11)^2} \leqslant s \leqslant s_{10} + \frac{1}{2(10)^2}$$

Using $s_{10} \approx 1.197532$, we get $1.201664 \leqslant s \leqslant 1.202532$

If we approximate *s* by the midpoint of this interval, then the error is at most half the length of the interval. So

$$\sum_{n=1}^\infty \frac{1}{n^3} \approx 1.2021 \qquad \text{with error} < 0.0005$$

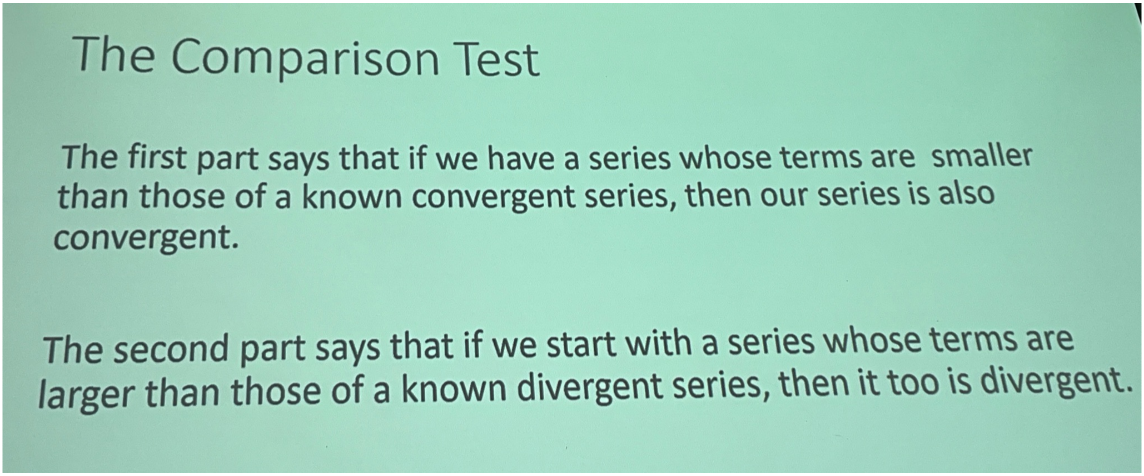


The Comparison Test

Suppose that Σa_n and Σb_n are series with positive terms.

(i) If Σb_n is convergent and $a_n \leqslant b_n$ for all n, then Σa_n is also convergent.

(ii) If Σb_n is divergent and $a_n \geqslant b_n$ for all n, then Σa_n is also divergent.



The Comparison Test

The first part says that if we have a series whose terms are smaller than those of a known convergent series, then our series is also convergent.

The second part says that if we start with a series whose terms are larger than those of a known divergent series, then it too is divergent.

Converge or Diverge = $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$

$$\downarrow$$

$$\frac{5}{2n^2+4n+3} < \frac{5}{2n^2}$$

We know = $\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} \longrightarrow$ converge

$$\downarrow$$

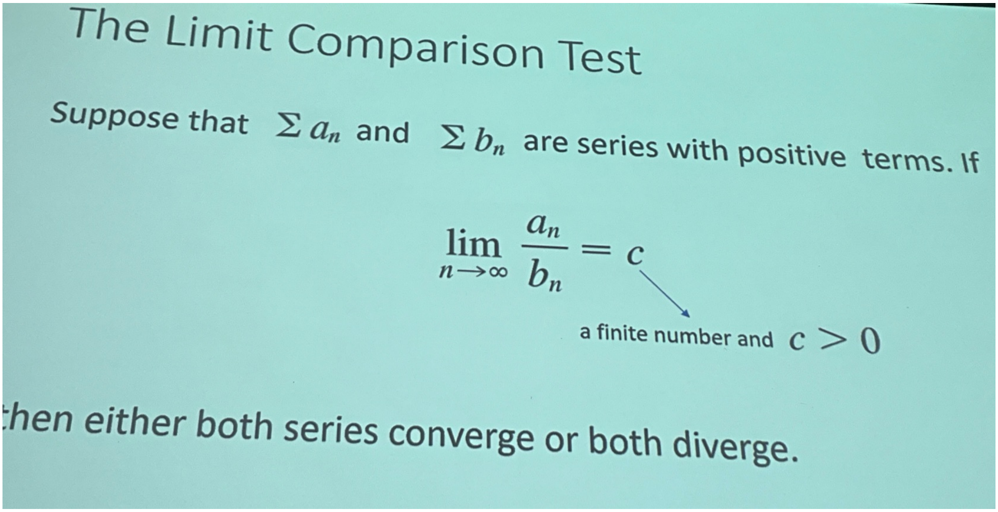
$p = 2 > 1$

$\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3} \longrightarrow$ Converge by part of the comparison test.

Converge or Diverge : $\sum_{k=1}^{\infty} \frac{\ln k}{k}$

$$\frac{\ln k}{k} > \frac{1}{k} \qquad k \geq 3$$

$\sum \frac{1}{k}$ is divergent (p series with p = 1) → Diverge by comparison test.



The Limit Comparison Test

Suppose that Σa_n and Σb_n are series with positive terms. If

$$\lim_{n \to \infty} \frac{a_n}{b_n} = c$$

a finite number and $c > 0$

then either both series converge or both diverge.

Converge or Diverge $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$

$$\frac{1}{2^n-1} > \frac{1}{2^n}$$

→ converge → thus, by the limit comparison test it also converges

$n - (n-1)$

$n - n + 1$

$$\frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} = \frac{2^n}{2^n-1} = \lim_{n \to \infty} = \frac{1}{1 - \frac{1}{2}^n} = 1 > 0$$

converge

Converge or Diverge. $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n5}}$

$\downarrow$

$a_n \quad \frac{2n^2+3n}{\sqrt{5+n^5}} = \rightarrow \quad b_n = \frac{2n^2}{n^{\frac{5}{2}}} = \frac{2}{n^{\frac{1}{2}}}$,

$= \lim_{n \to \infty} \frac{a_n}{b_n} = \frac{2n^2+3n}{\sqrt{5+n^5}} \times \frac{n^{\frac{1}{2}}}{2} = \frac{2n^{\frac{5}{2}} + 3n^{\frac{3}{2}}}{2\sqrt{5+n^5}}$

$$= \frac{\frac{2n^{\frac{5}{2}}}{n^{\frac{5}{2}}} + \frac{3n^{\frac{3}{2}}}{n^{\frac{5}{2}}}}{2\sqrt{\frac{5}{n^5} + \frac{n^5}{n^5}}} = \frac{2+0}{2\sqrt{0+1}} = 1$$

$1 > 0$ Prove by limit comparison test,

as bn $\lim_{n \to \infty} \frac{2}{n^{\frac{1}{2}}}$ = Diverge (P series < 1)

$\downarrow$

Thus a_n also converge. #

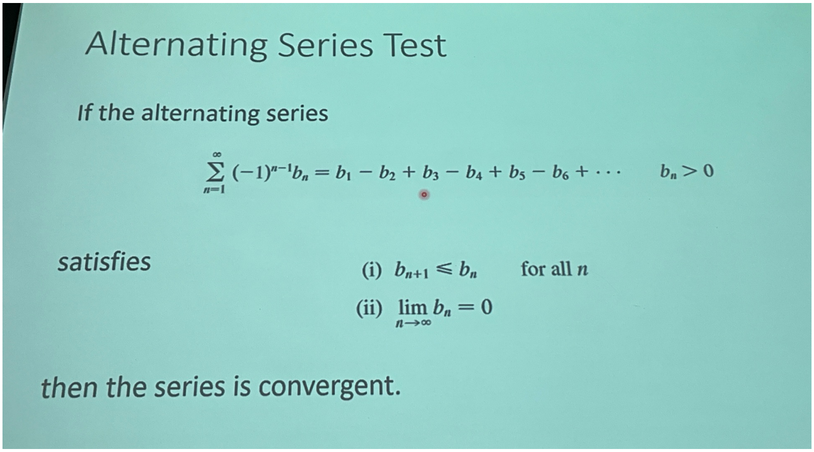
Alternating Series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}\ \cdots = \sum_{n=1}^{\infty} (-1)^{n-1}\ \frac{1}{n}$$

$$-\frac{1}{2}+\frac{2}{3}-\frac{3}{4}+\frac{4}{5}-\frac{5}{6}+\frac{6}{7}\ \cdots = \sum_{n=1}^{\infty} (-1)^{n}\ \frac{n}{n+1}$$

$a_n = (-1)^{n-1} b_n$ or $a_n = (-1)^n b_n$

B_n is a positive number



Alternating Series Test

If the alternating series

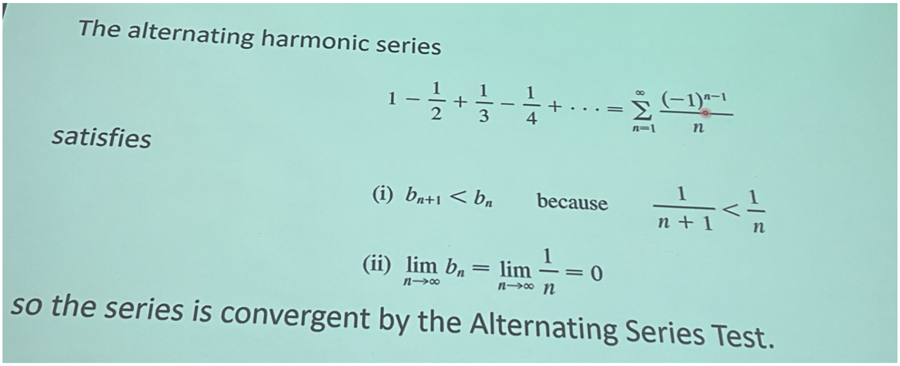
$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad b_n > 0$$

satisfies

(i) $b_{n+1} \leqslant b_n$ for all n

(ii) $\lim_{n\to\infty} b_n = 0$

then the series is convergent.



The alternating harmonic series

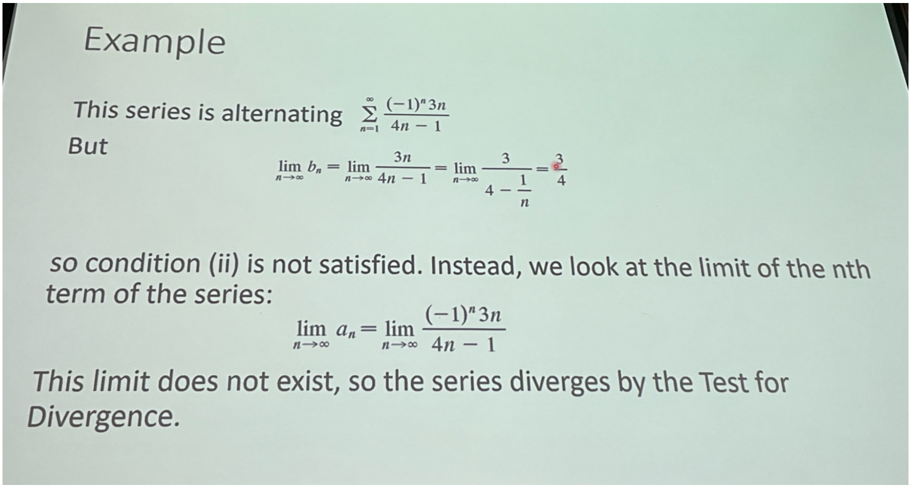
$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

satisfies

(i) $b_{n+1} < b_n$ because $\frac{1}{n+1} < \frac{1}{n}$

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$

so the series is convergent by the Alternating Series Test.



Example

This series is alternating $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$

But

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3n}{4n-1} = \lim_{n\to\infty} \frac{3}{4-\frac{1}{n}} = \frac{3}{4}$$

so condition (ii) is not satisfied. Instead, we look at the limit of the nth term of the series:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n 3n}{4n-1}$$

This limit does not exist, so the series diverges by the Test for Divergence.

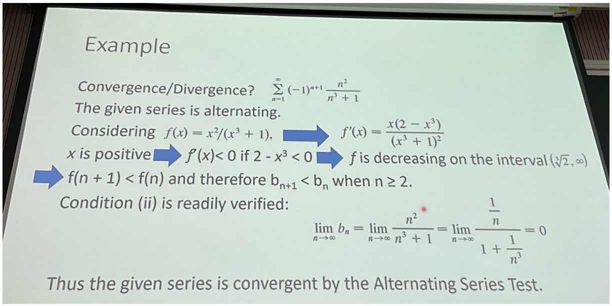
Convergence / Divergence.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$$

$\frac{1}{2}$, $\frac{4}{9}$, $\frac{9}{28}$

① ✓ $b_{n+1} < b_n$

$$\lim_{n\to\infty} \frac{n^2}{n^3+1} = \frac{\frac{1}{n}}{1+\frac{1}{n^3}} = \frac{0}{1} \longrightarrow \text{Convergent.}$$



Example

Convergence/Divergence? $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$

The given series is alternating.

Considering $f(x) = x^2/(x^3+1)$, ➡ $f'(x) = \frac{x(2-x^3)}{(x^3+1)^2}$

x is positive ➡ $f'(x) < 0$ if $2 - x^3 < 0$ ➡ f is decreasing on the interval $(\sqrt[3]{2}, \infty)$

➡ $f(n+1) < f(n)$ and therefore $b_{n+1} < b_n$ when $n \geq 2$.

Condition (ii) is readily verified:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2}{n^3+1} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n^3}} = 0$$

Thus the given series is convergent by the Alternating Series Test.

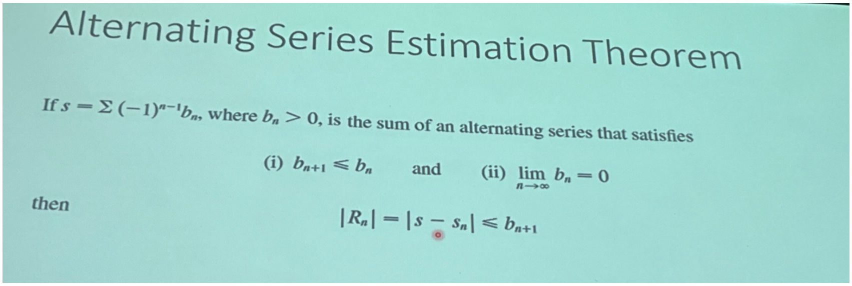
1, 2 = Application of interval

3, ~~4~~ = find integral

4 = Application integral

5 = Probability , 6-7, = Series

Theorem



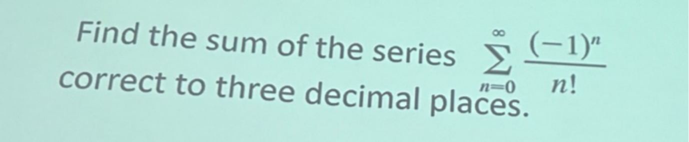
Alternating Series Estimation Theorem

If $s = \Sigma (-1)^{n-1} b_n$, where $b_n > 0$, is the sum of an alternating series that satisfies

$$\text{(i) } b_{n+1} \leqslant b_n \quad \text{and} \quad \text{(ii) } \lim_{n \to \infty} b_n = 0$$

then

$$|R_n| = |s - s_n| \leqslant b_{n+1}$$



Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ correct to three decimal places.

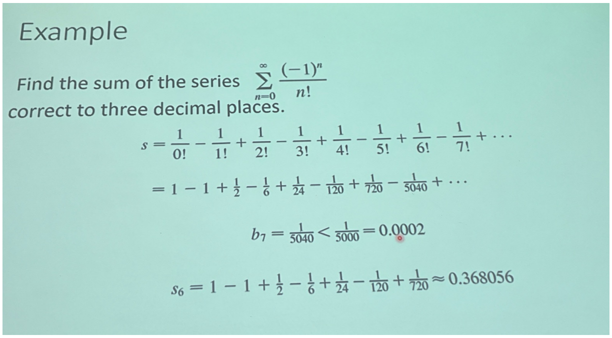
(i) $b_{n+1} \leq b_n$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \rightarrow b_n = \frac{1}{n!}$$

$$\downarrow$$

$$\frac{1}{(n+1)!} < \frac{1}{n!}$$

(ii) $0 < \frac{1}{n!} < \frac{1}{n} \rightarrow 0$ so $\frac{1}{n!} \rightarrow 0$ as $n \rightarrow \infty$



Example

Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ correct to three decimal places.

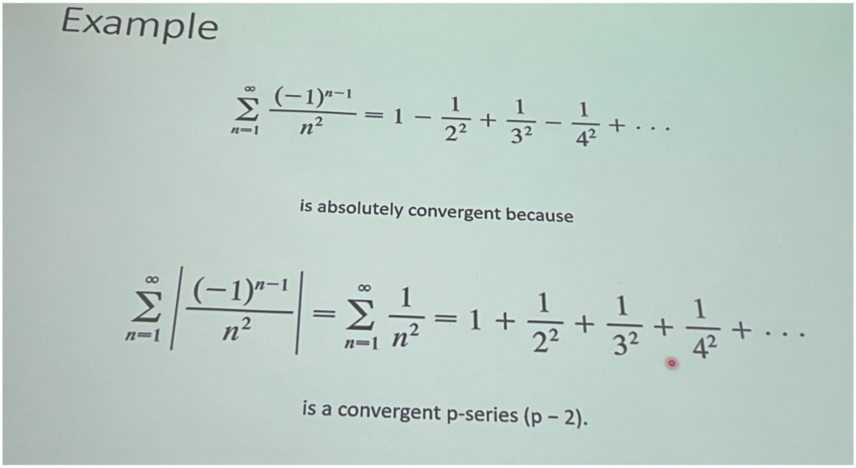
$$s = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \cdots$$

$$= 1 - 1 + \tfrac{1}{2} - \tfrac{1}{6} + \tfrac{1}{24} - \tfrac{1}{120} + \tfrac{1}{720} - \tfrac{1}{5040} + \cdots$$

$$b_7 = \tfrac{1}{5040} < \tfrac{1}{5000} = 0.0002$$

$$s_6 = 1 - 1 + \tfrac{1}{2} - \tfrac{1}{6} + \tfrac{1}{24} - \tfrac{1}{120} + \tfrac{1}{720} \approx 0.368056$$

Absolute convergence



Example

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

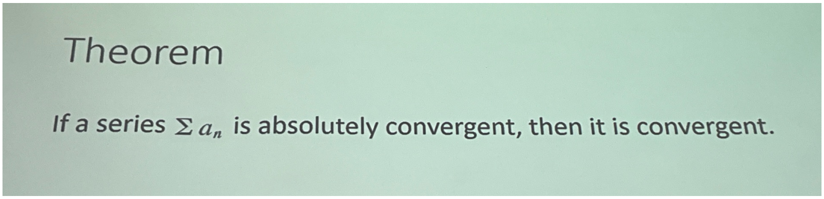
is absolutely convergent because

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

is a convergent p-series (p – 2).

Conditionally Converget

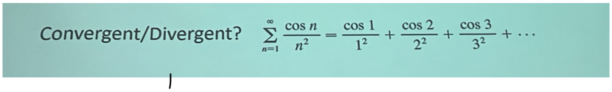
Convergent but not absolutely convergent.



Theorem

If a series Σa_n is absolutely convergent, then it is convergent.

Ex.



Convergent/Divergent? $\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \cdots$

↓

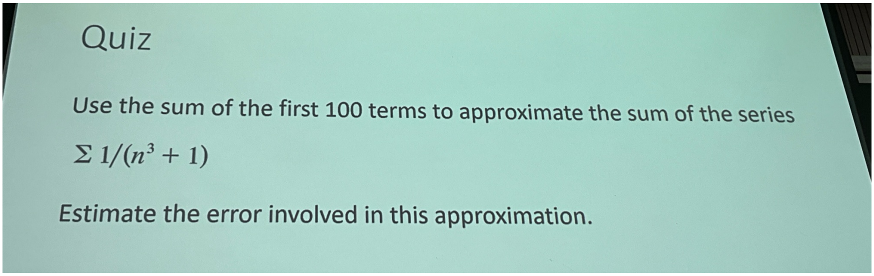
not alternate

$$\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$$

$$|\cos n| \le 1 \longrightarrow \frac{|\cos n|}{n^2} \le \frac{1}{n^2} \longrightarrow \sum \frac{1}{n^2} \text{ is convergent}$$

– p series with $p = 2$

↓

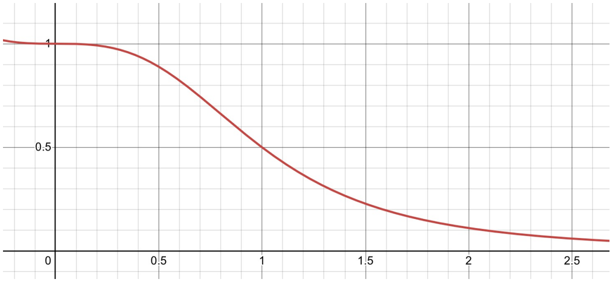


Quiz

Use the sum of the first 100 terms to approximate the sum of the series

$\Sigma\, 1/(n^3 + 1)$

Estimate the error involved in this approximation.



Decrease

$$\int_n^\infty \frac{1}{x^3+1}\,dx = \lim_{t\to\infty}\left[\frac{1}{3}\ln|x+1| - \frac{1}{6}\ln(x^2-x+1) + \frac{\sqrt{3}}{3}\arctan\left(\frac{2\sqrt{3}}{3}x - \frac{\sqrt{3}}{3}\right)\right]_n^t$$

$$= \left[\frac{1}{3}\ln|t+1| - \cancel{\frac{1}{6}\ln(t^2-t+1) + \frac{\sqrt{3}}{3}\arctan\left(\frac{2\sqrt{3}}{3}t - \frac{\sqrt{3}}{3}\right)}\right] -$$

$$\frac{1}{3}\ln|n+1| - \cancel{\frac{1}{6}\ln(n^2-n+1) + \frac{\sqrt{3}}{3}\arctan\left(\frac{2\sqrt{3}}{3}n - \frac{\sqrt{3}}{3}\right)}$$

---

as $\frac{1}{n^3+1} < \frac{1}{n^3}$ comparison test

$\longrightarrow$ (pseries as $p > 1$) so converge

$$= \int_n^\infty \frac{1}{n^3} = \lim_{t\to\infty}\left[-\frac{1}{2x^2}\right]_n^t = \lim_{t\to\infty}\left[-\underset{\substack{\downarrow\\0}}{\frac{1}{2t^2}} + \frac{1}{2n^2}\right]$$

$$= \frac{1}{2n^2}$$

First 100 term = $\frac{1}{2(100)^2}$ = 0.00005 → max error

approximate : $S_n + \int_{n+1}^{\infty} f(x)\,dx \le S \le S_n + \int_{n}^{\infty} f(x)\,dx$

ifinite series

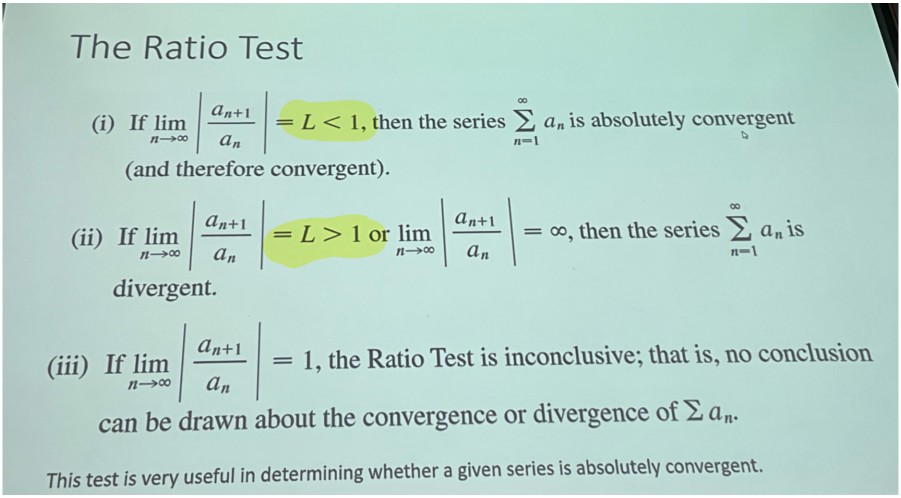
$$S_n + \frac{1}{2(n+1)^2} \le S \le S_n + \frac{1}{2n^2}$$

$$S_{100} + \frac{1}{2(101)^2} \le S \le S_{100} + \frac{1}{2(100)^2}$$

S_{100} = 0.6864538 + → answers (computer)

Ratio test

Thus $\sum \frac{|\cos n|}{n2} \to$ convergent.



The Ratio Test

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

This test is very useful in determining whether a given series is absolutely convergent.

Example

Test the for absolute convergence. $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$

$$\lim_{n\to\infty} \frac{\frac{(n+1)^3}{3^{n+1}}}{\frac{n^3}{3^n}}$$

$$\downarrow$$

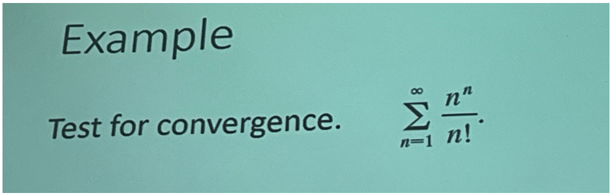
$$= \frac{(n+1)^3}{3^{n+1}} \times \frac{3^n}{n^3}$$

$$= \frac{(n+1)^3}{3n^3} = \lim_{n\to\infty} \frac{1}{3}\left(\frac{n+1}{n}\right)^3$$

$$= \frac{1}{3}\left(1+\frac{1}{n}\right)^3 \to \frac{1}{3} < 1$$

$\uparrow$

converge ratiotest



Example

Test for convergence. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

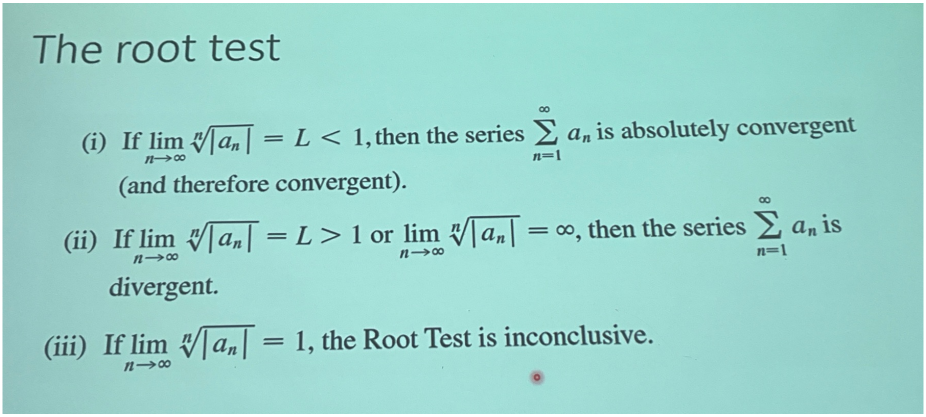
$$\lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)\cancel{!}} \times \frac{\cancel{n!}}{n^n}$$

$$= \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \to e$$

✓

$e > 1$, so diverge.

<u>Root test</u>

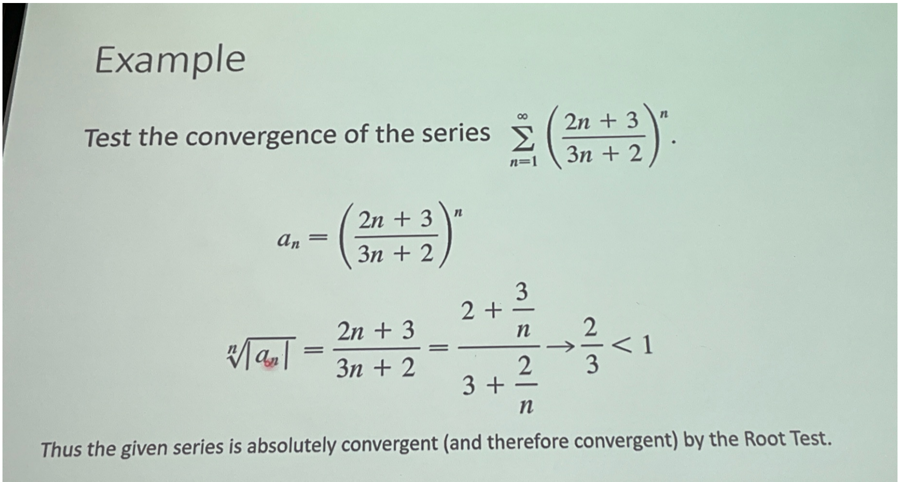


The root test

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.



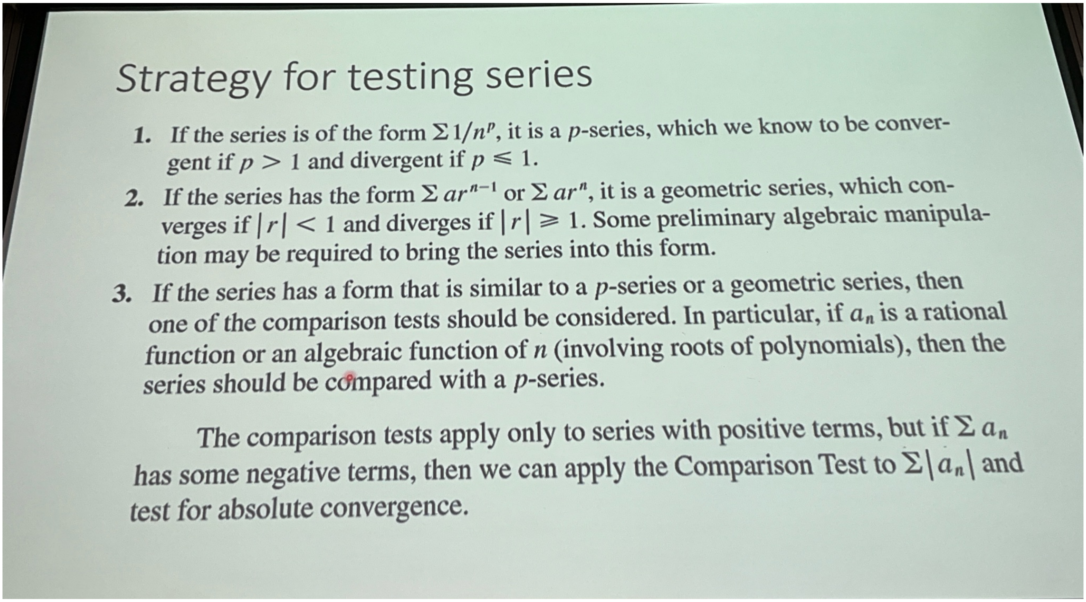
Example

Test the convergence of the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$.

$$a_n = \left(\frac{2n+3}{3n+2}\right)^n$$

$$\sqrt[n]{|a_n|} = \frac{2n+3}{3n+2} = \frac{2+\frac{3}{n}}{3+\frac{2}{n}} \to \frac{2}{3} < 1$$

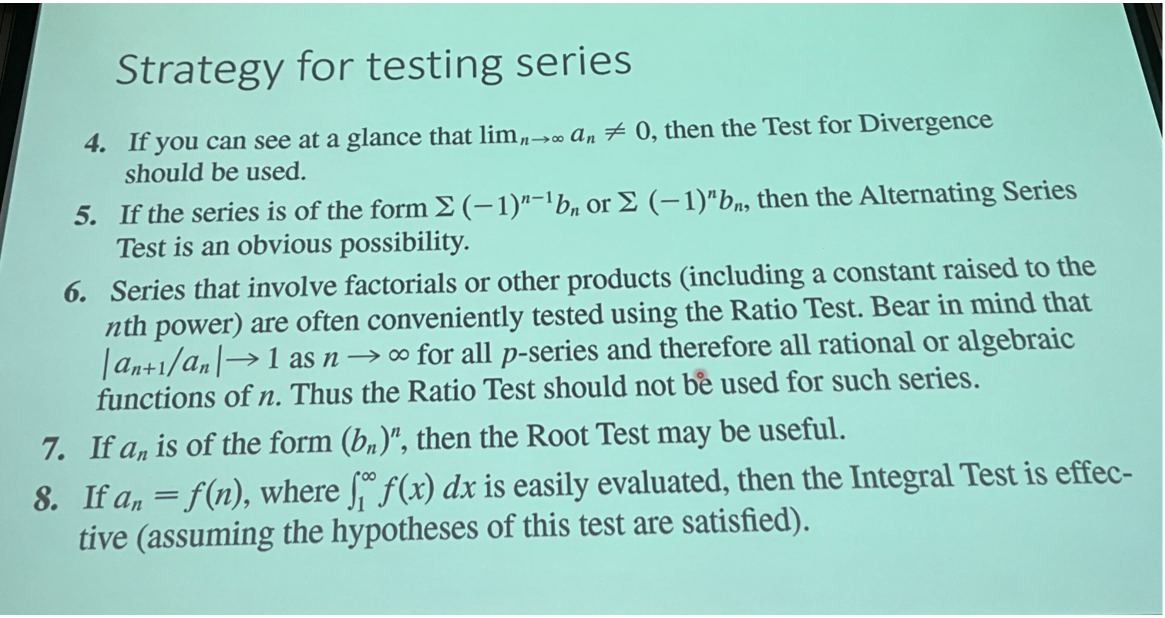
Thus the given series is absolutely convergent (and therefore convergent) by the Root Test.



Strategy for testing series

1. If the series is of the form $\Sigma\, 1/n^p$, it is a p-series, which we know to be convergent if $p > 1$ and divergent if $p \leqslant 1$.
2. If the series has the form $\Sigma\, ar^{n-1}$ or $\Sigma\, ar^n$, it is a geometric series, which converges if $|r| < 1$ and diverges if $|r| \geqslant 1$. Some preliminary algebraic manipulation may be required to bring the series into this form.
3. If the series has a form that is similar to a p-series or a geometric series, then one of the comparison tests should be considered. In particular, if a_n is a rational function or an algebraic function of n (involving roots of polynomials), then the series should be compared with a p-series.

The comparison tests apply only to series with positive terms, but if $\Sigma\, a_n$ has some negative terms, then we can apply the Comparison Test to $\Sigma\, |a_n|$ and test for absolute convergence.



Strategy for testing series

4. If you can see at a glance that $\lim_{n \to \infty} a_n \neq 0$, then the Test for Divergence should be used.
5. If the series is of the form $\Sigma\, (-1)^{n-1} b_n$ or $\Sigma\, (-1)^n b_n$, then the Alternating Series Test is an obvious possibility.
6. Series that involve factorials or other products (including a constant raised to the nth power) are often conveniently tested using the Ratio Test. Bear in mind that $|a_{n+1}/a_n| \to 1$ as $n \to \infty$ for all p-series and therefore all rational or algebraic functions of n. Thus the Ratio Test should not be used for such series.
7. If a_n is of the form $(b_n)^n$, then the Root Test may be useful.
8. If $a_n = f(n)$, where $\int_1^\infty f(x)\, dx$ is easily evaluated, then the Integral Test is effective (assuming the hypotheses of this test are satisfied).

Example

$$\sum_{n=1}^{\infty} \frac{n-1}{2n+1} \rightarrow \lim_{n\to\infty} \frac{1-\frac{1}{n}}{2+\frac{1}{n}} = \frac{1}{2} \neq 0$$

↓

Diverge.

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2} \rightarrow$$

→ Comparison test = $\frac{n^{\frac{3}{2}}}{3n^{3}} = \frac{1}{3} \lim_{n\to\infty} \frac{1}{n^{\frac{3}{2}}}$

↓

$p > 1$ so converge

↓

After that use $\sum_{n=0}^{\infty} \frac{a_n}{b_n} \rightarrow$ (limit Comparison test.)

$$= \frac{\sqrt{n^3+1}}{3n^3+4n^2+2} \cdot 3n^{\frac{3}{2}}$$

$$= \frac{3n^{\frac{3}{2}}\sqrt{n^3+1}}{3n^3+4n^2+2}$$

$$= \frac{3n^{\frac{3}{2}}\sqrt{n^3\left(1+\frac{1}{n^3}\right)}}{3n^3+4n^2+2}$$

$$= \frac{3n^{\frac{3}{2}} \cdot n^{\frac{3}{2}}\sqrt{1+\frac{1}{n^3}}}{3n^3+4n^2+2}$$

$$= \frac{3n^3\sqrt{1+\frac{1}{n^3}}}{3n^3+4n^2+2}$$

$$= \frac{3\sqrt{1+\frac{1}{n^3}}}{3+\frac{4}{n}+\frac{2}{n^3}}$$

$$= \frac{3}{3} = 1 > 0$$

↓

So both converge

$$\sum_{n=1}^{\infty} n e^{-n^2} \to$$ Integral test

$$x e^{-x2}$$

$$\int_1^{\infty} x e^{-x2} \, d\cancel{x}$$

$$v = -x^2 \qquad dv = -2x \, dx$$

$$dx = \frac{dv}{-2x}$$

$$= \int_1^{\infty} \cancel{x} \, e^{v} \cdot \frac{dv}{-2\cancel{x}}$$

$$= -\frac{1}{2} \int_1^{\infty} e^{v} \, dv$$

sub v

$$= -\frac{1}{2} \left[\lim_{t \to \infty} e^{-x^2} \Big|_1^t \right] = -\frac{1}{2} \left[\lim_{t \to \infty} \overset{0}{e^{-(t)^2}} - e^{-(1)^2} \right]$$

≈ 0.1839 so convergd

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4+1} \to \text{Alternating}$$

$$b_{n+1} < b_n \quad ✓$$

$$\lim_{n\to\infty} \frac{n^3}{n^4+1} = \frac{\frac{1}{n}}{1+\frac{1}{n^4}} = 0 \quad \text{converge}$$

$$\sum_{k=1}^{\infty} \frac{2^k}{k!} = \text{Ratio test.}$$

$$= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{2^{k+1}}{(k+1)\cancel{!}} \cdot \frac{\cancel{k!}}{2K} = \frac{2}{K+1}$$

$$\lim_{K\to\infty} \frac{\frac{2}{K}}{1+\frac{1}{K}} = 0 \to < 1$$

converge

$$\sum_{n=1}^{\infty} \frac{1}{2+3^n} = \longrightarrow \frac{1}{2+3^n} < \frac{1}{3^n} \to b_n$$

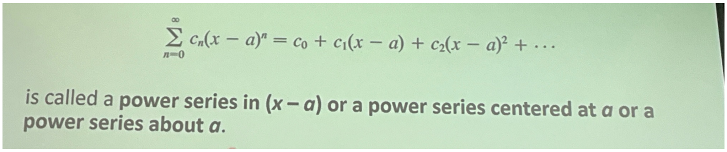
$$\downarrow$$

$$a = \frac{1}{3} \quad r = \frac{1}{3} \longrightarrow |r| < 1$$

so converge

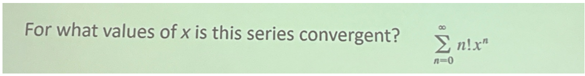
according to comparison test, as b_n converge, a_n also converge

Power Series



$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

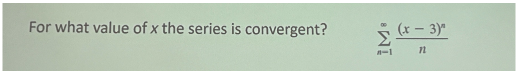
is called a **power series in (*x* – *a*) or a power series centered at *a* or a power series about *a*.**



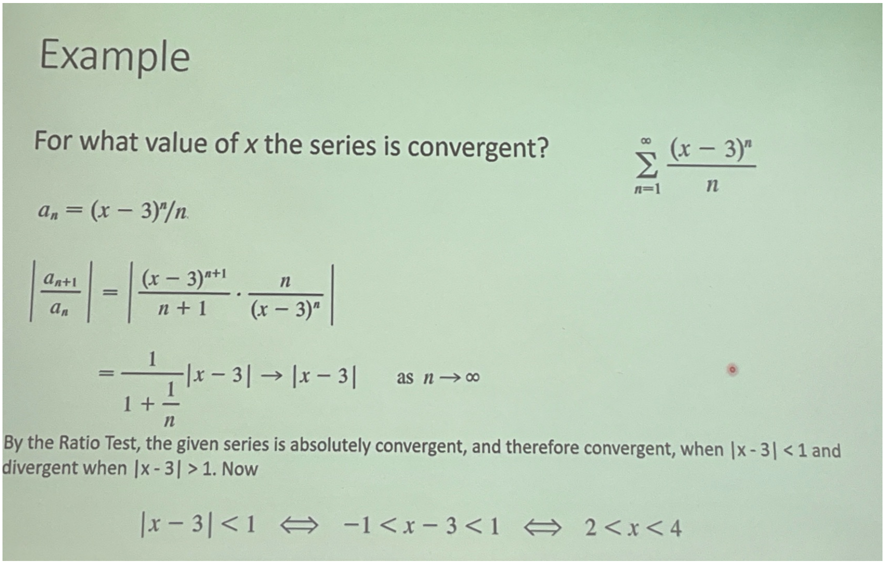
For what values of *x* is this series convergent? $\sum_{n=0}^{\infty} n!x^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)\cancel{!}x^{n+1}}{\cancel{n}!\,x^n}\right| = \lim_{n\to\infty}(n+1)|x| = \infty$$

Converge when $x = 0$.



For what value of *x* the series is convergent? $\sum_{n=1}^{\infty}\frac{(x-3)^n}{n}$



Example

For what value of *x* the series is convergent? $\sum_{n=1}^{\infty}\frac{(x-3)^n}{n}$

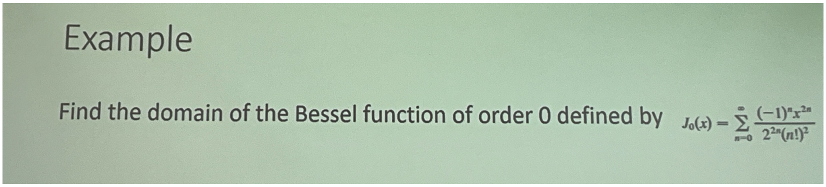
$a_n = (x-3)^n/n.$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-3)^{n+1}}{n+1}\cdot\frac{n}{(x-3)^n}\right|$$

$$= \frac{1}{1+\frac{1}{n}}|x-3| \to |x-3| \quad \text{as } n\to\infty$$

By the Ratio Test, the given series is absolutely convergent, and therefore convergent, when |x - 3| < 1 and divergent when |x - 3| > 1. Now

$$|x-3|<1 \iff -1<x-3<1 \iff 2<x<4$$



Example

Find the domain of the Bessel function of order 0 defined by $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$

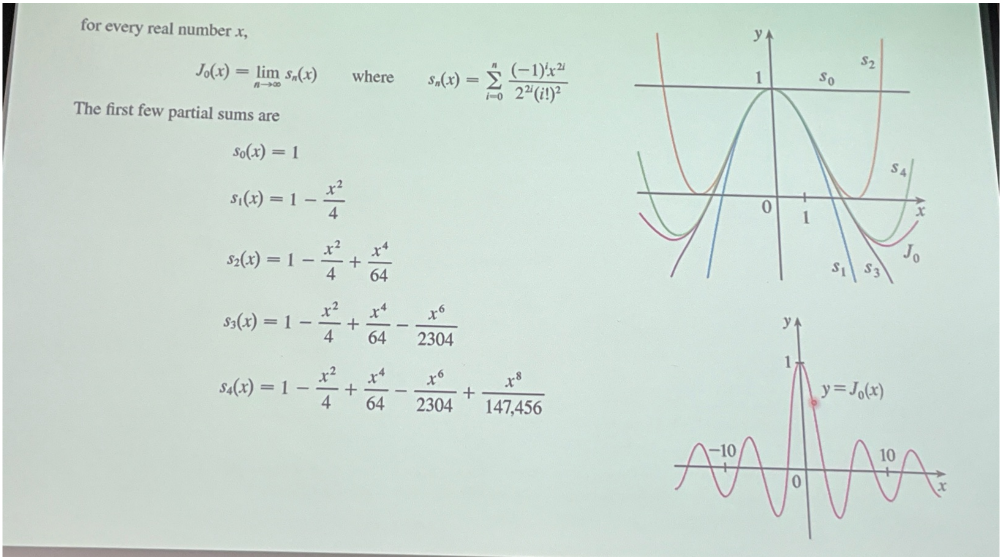
$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

$$\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+1} x^{2(n+1)}}{2^{2(n+1)}((n+1)!)^2} \cdot \frac{2^{2n}(n!)^2}{(-1)^n x^{2n}}$$

$$= \frac{x^{2n+2}}{2^{2n+2}(n+1)^2(n!)^2} \cdot \frac{2^{2n}(n!)^2}{x^{2n}}$$

$$= \frac{x^2}{4(n+1)^2} \to 0 < 1$$ for all x $\forall$

J_0 is $(-\infty, \infty) = \mathbb{R}$.



for every real number x,

$$J_0(x) = \lim_{n\to\infty} s_n(x) \quad \text{where} \quad s_n(x) = \sum_{i=0}^{n} \frac{(-1)^i x^{2i}}{2^{2i}(i!)^2}$$

The first few partial sums are

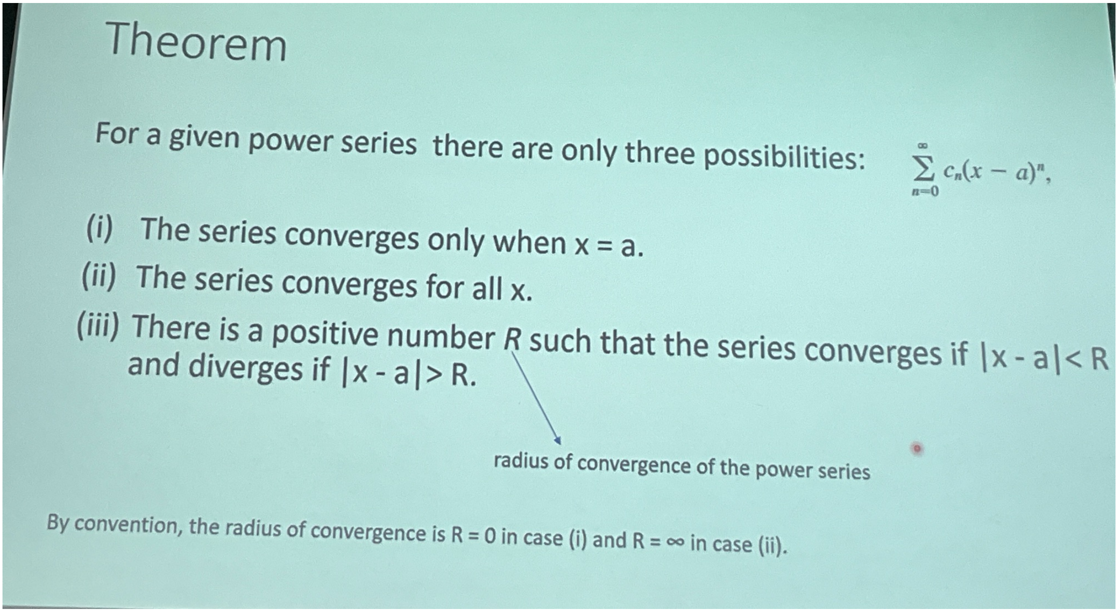
$$s_0(x) = 1$$

$$s_1(x) = 1 - \frac{x^2}{4}$$

$$s_2(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64}$$

$$s_3(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304}$$

$$s_4(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \frac{x^8}{147{,}456}$$



Theorem

For a given power series there are only three possibilities: $\sum_{n=0}^{\infty} c_n(x-a)^n,$

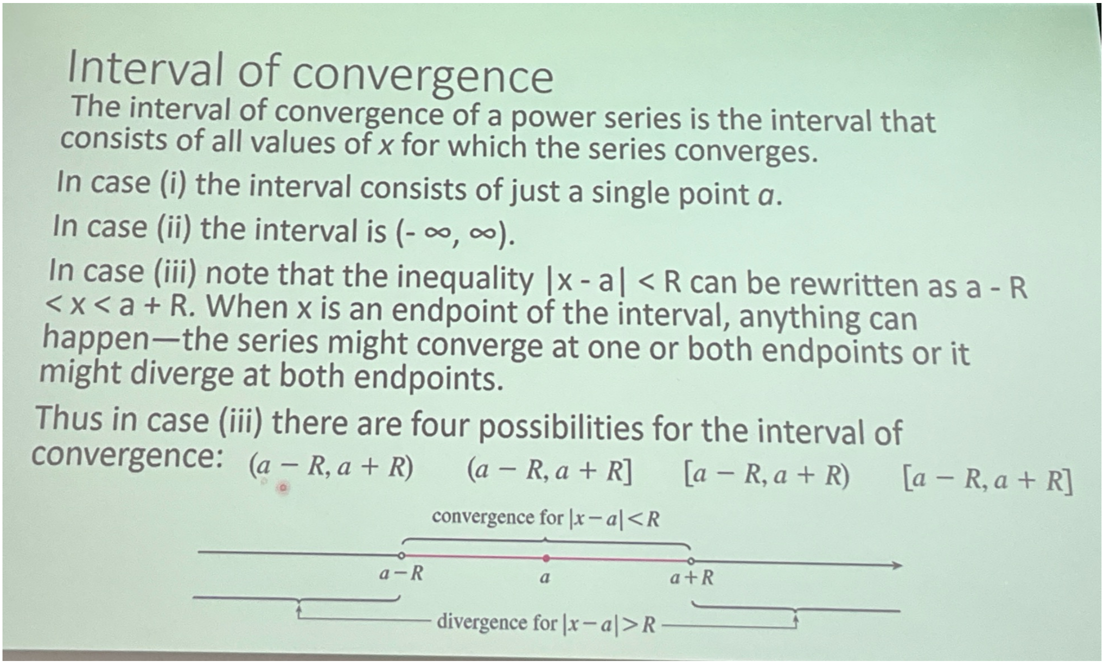
(i) The series converges only when x = a.

(ii) The series converges for all x.

(iii) There is a positive number *R* such that the series converges if |x - a|< R and diverges if |x - a|> R.

radius of convergence of the power series

By convention, the radius of convergence is R = 0 in case (i) and R = ∞ in case (ii).



Interval of convergence

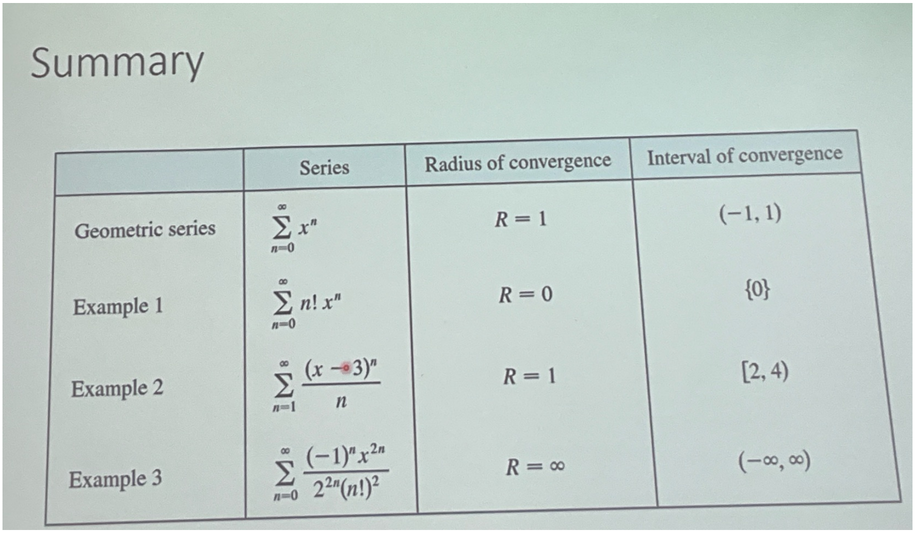
The interval of convergence of a power series is the interval that consists of all values of *x* for which the series converges.

In case (i) the interval consists of just a single point *a*.

In case (ii) the interval is (- ∞, ∞).

In case (iii) note that the inequality |x - a| < R can be rewritten as a - R < x < a + R. When x is an endpoint of the interval, anything can happen—the series might converge at one or both endpoints or it might diverge at both endpoints.

Thus in case (iii) there are four possibilities for the interval of convergence: $(a-R, a+R)$ $(a-R, a+R]$ $[a-R, a+R)$ $[a-R, a+R]$

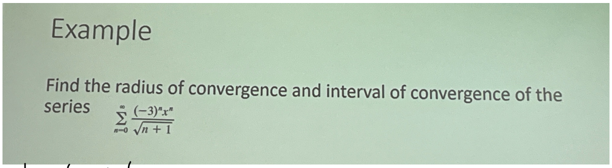


Summary

	Series	Radius of convergence	Interval of convergence
Geometric series	$\sum_{n=0}^{\infty} x^n$	$R = 1$	$(-1, 1)$
Example 1	$\sum_{n=0}^{\infty} n!\, x^n$	$R = 0$	$\{0\}$
Example 2	$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$	$R = 1$	$[2, 4)$
Example 3	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$	$R = \infty$	$(-\infty, \infty)$

Radius of Convergence

Usually use ratio test



Example

Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

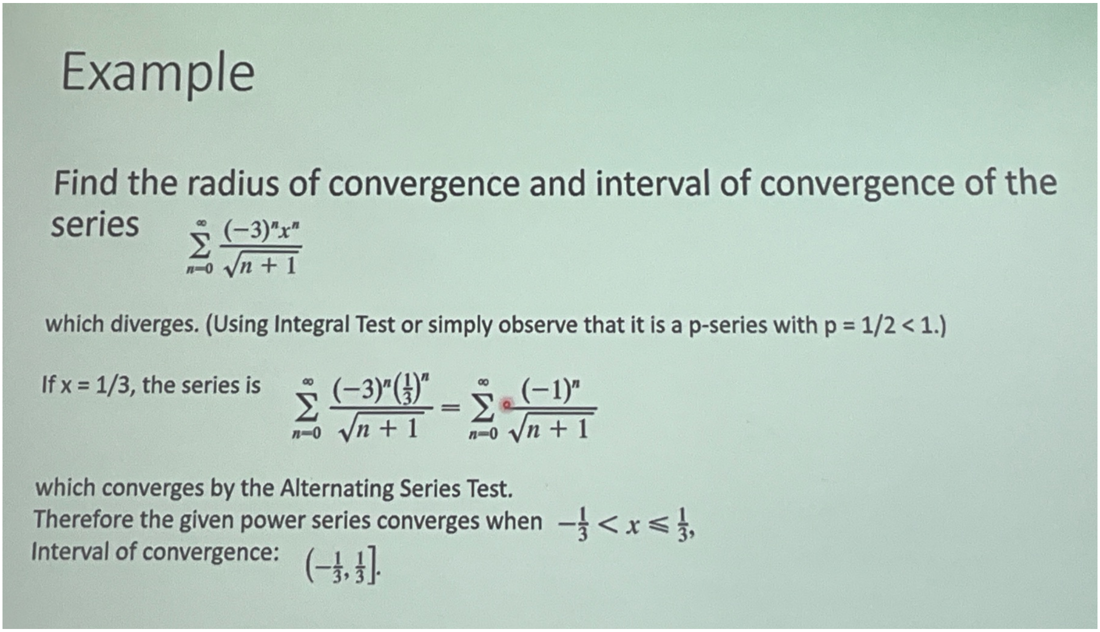
$$= \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n} = \left| -3x \sqrt{\frac{n+1}{n+2}} \right|$$

$$= 3 \lim_{n\to\infty} \sqrt{\frac{n+1}{n+2}}\, |x| = 3 \lim_{n\to\infty} \sqrt{\frac{1+\frac{1}{2}}{1+\frac{2}{n}}}\, |x|$$

$$= 3|x| \text{ as } n \to \infty$$

↓

Converge when $3|x| < 1$ and Diverge when $3|x| > 1$



Example

Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$

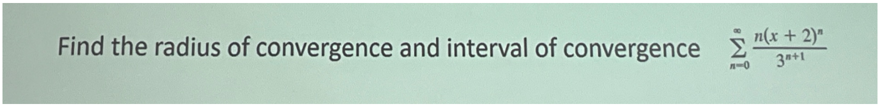
which diverges. (Using Integral Test or simply observe that it is a p-series with p = 1/2 < 1.)

If x = 1/3, the series is $\sum_{n=0}^{\infty} \frac{(-3)^n(\frac{1}{3})^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$

which converges by the Alternating Series Test.

Therefore the given power series converges when $-\frac{1}{3} < x \leqslant \frac{1}{3}$,

Interval of convergence: $(-\frac{1}{3}, \frac{1}{3}]$.



Find the radius of convergence and interval of convergence $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$

$$\sum_{n=0}^{\infty} \frac{(n+1)(x+2)^{\cancel{n+1}}}{3^{\cancel{n+2}}} \times \frac{\cancel{3^{n+1}}}{n\,\cancel{(x+2)^n}}$$

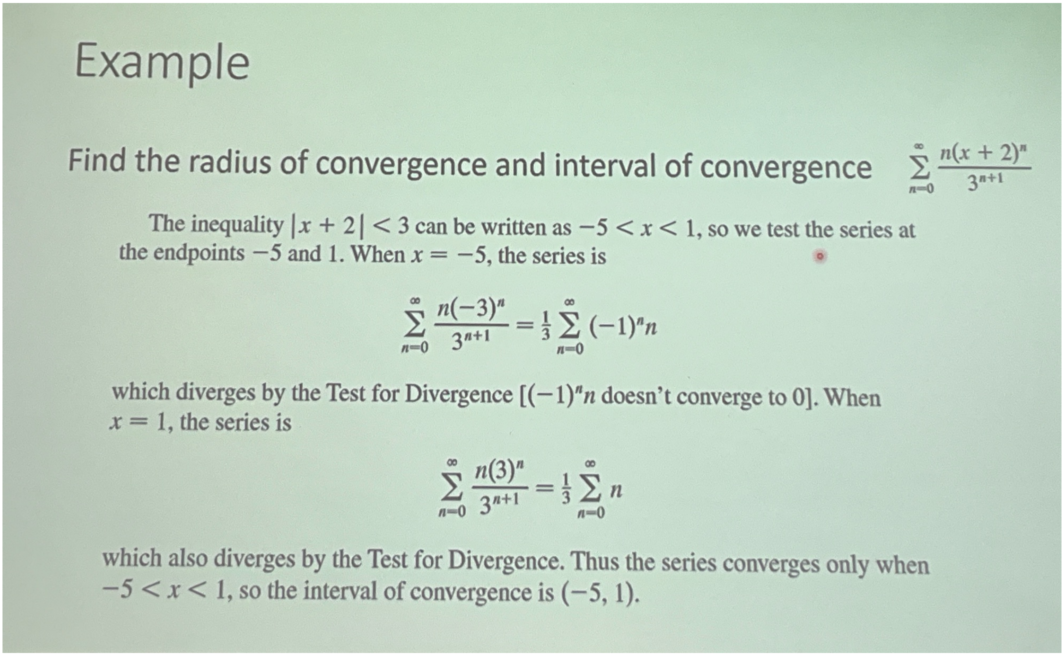
$$= \frac{(n+1)(x+2)}{3n}$$

$$\frac{n+1}{n} = 1+\frac{1}{n} \rightarrow \lim_{n\to\infty} 1+\frac{1}{n} = 1$$

$$= \frac{1}{3}\lim_{n\to\infty} \frac{(n+1)(x+2)}{n} = \rightarrow \frac{1}{3}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)(x+2)$$

$$= \frac{|x+2|}{3} < 1 \rightarrow \text{Converge}$$

$$\frac{|x+2|}{3} > 1 \rightarrow \text{Diverge}$$



Example

Find the radius of convergence and interval of convergence $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$

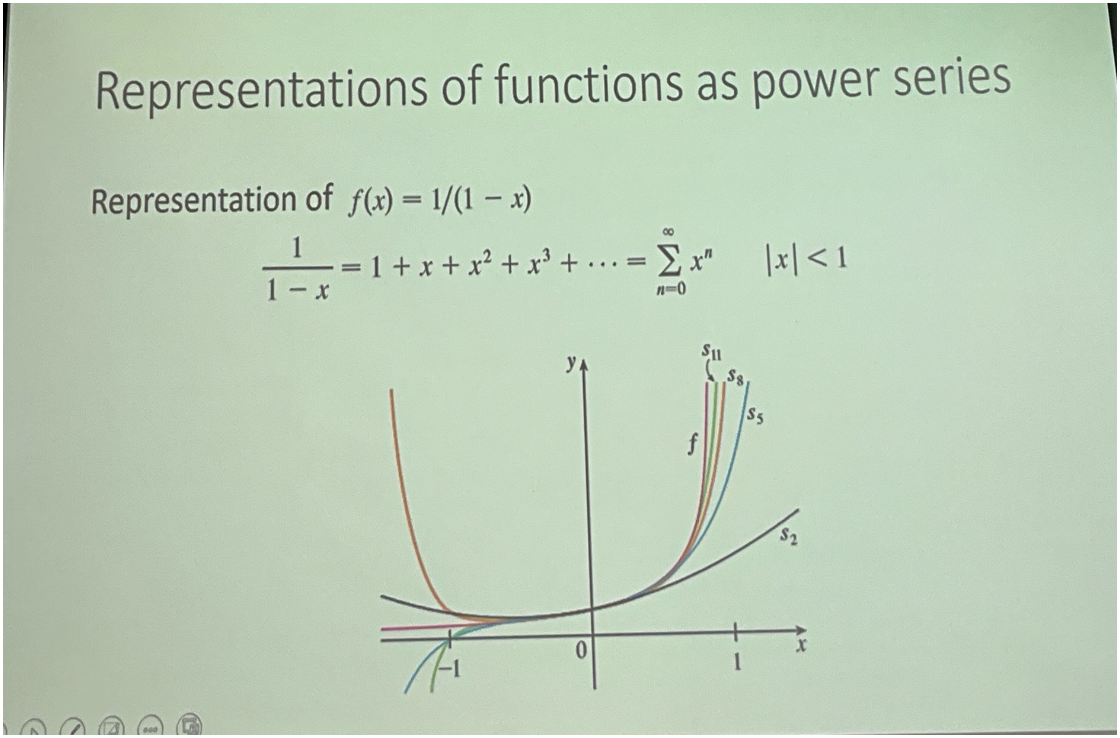
The inequality $|x+2| < 3$ can be written as $-5 < x < 1$, so we test the series at the endpoints -5 and 1. When $x = -5$, the series is

$$\sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n n$$

which diverges by the Test for Divergence [$(-1)^n n$ doesn't converge to 0]. When $x = 1$, the series is

$$\sum_{n=0}^{\infty} \frac{n(3)^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} n$$

which also diverges by the Test for Divergence. Thus the series converges only when $-5 < x < 1$, so the interval of convergence is $(-5, 1)$.



Representations of functions as power series

Representation of $f(x) = 1/(1-x)$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$$

Express $\frac{1}{1+x^2}$ as the sum of a power series

$$= f(x)\ \frac{1}{1+x^2} = \frac{a}{1-r}$$

$$= \frac{1}{1-(-x^2)} \longrightarrow \begin{array}{l} a = 1 \\ r = -x^2 \end{array}$$

$$= \sum_{n=0}^{\infty} 1(-x^2)^n$$

$r = -x^2$

$|r| < 1 \longrightarrow$ Converge

$|-x^2| < 1$

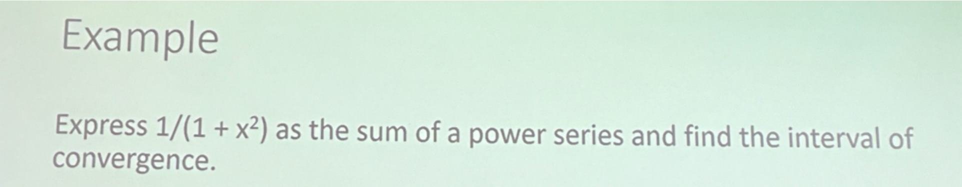
$x^2 < 1 \qquad x < 1$

$-1 < x \boxed{<1} \longrightarrow$ radius of convergence

Interval: $(-1, 1)$

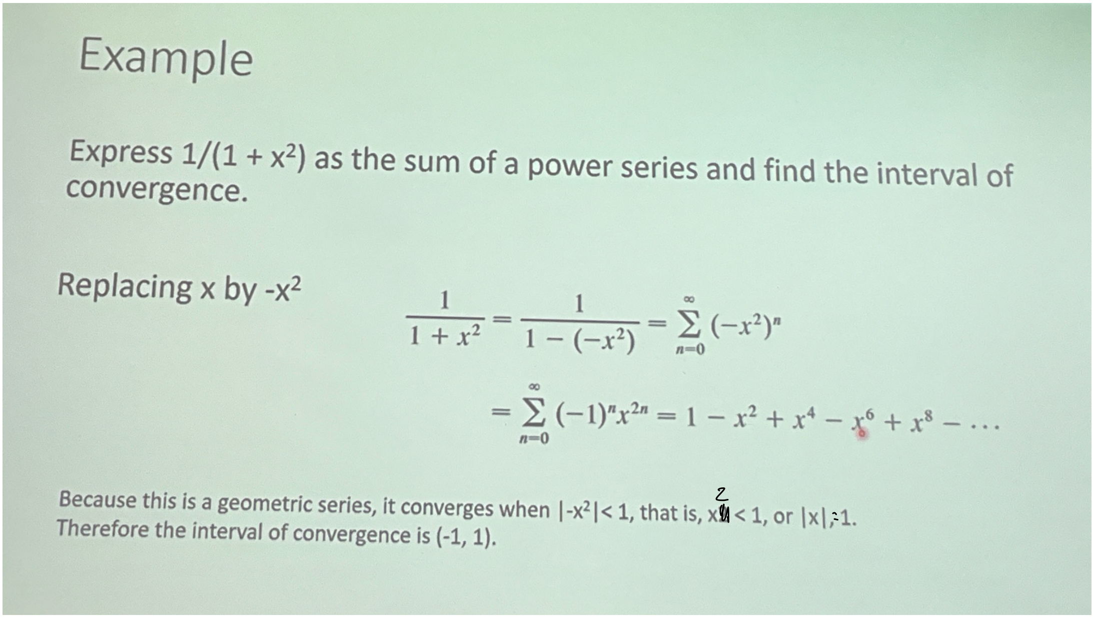
Sum of power series

$$= \sum_{n=0}^{\infty} (-1)^n (x^{2n})$$



Example

Express 1/(1 + x²) as the sum of a power series and find the interval of convergence.



Example

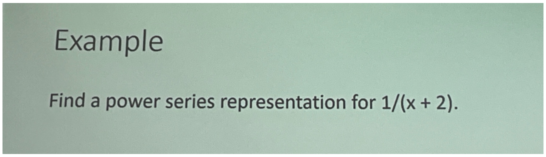
Express 1/(1 + x²) as the sum of a power series and find the interval of convergence.

Replacing x by -x²

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 - \dots$$

Because this is a geometric series, it converges when $|-x^2| < 1$, that is, $x^{\overset{2}{4}} < 1$, or $|x| < 1$. Therefore the interval of convergence is (-1, 1).



Example

Find a power series representation for 1/(x + 2).

Check later

!!

$$\frac{1}{x+2} = \frac{1}{2\left(1+\frac{x}{2}\right)} = \frac{\frac{1}{2}}{1+\frac{x}{2}} = \frac{\frac{1}{2}}{1-\left(-\frac{x}{2}\right)} = \frac{a}{1-r}$$

$a = \frac{1}{2}$ $r = -\frac{x}{2}$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)\left[\left(-\frac{x}{2}\right)\right]^n = \frac{1}{2^1} \cdot \frac{(-x)^n}{2^n}$$

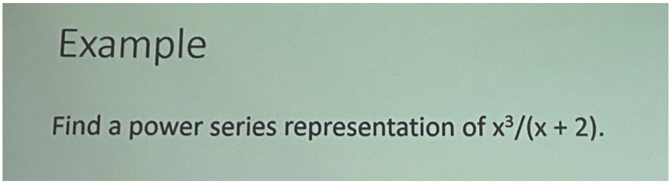
$$\sum_{n=0}^{\infty} \frac{(-1)^n (x)^n}{2^{n+1}}$$

$|r| < 1 \rightarrow$ converge

$\left|-\frac{x}{2}\right| < 1 \rightarrow x < 2$

$-2 < x \boxed{< 2} \rightarrow R$

Interval = $(-2, 2)$



Example

Find a power series representation of $x^3/(x + 2)$.

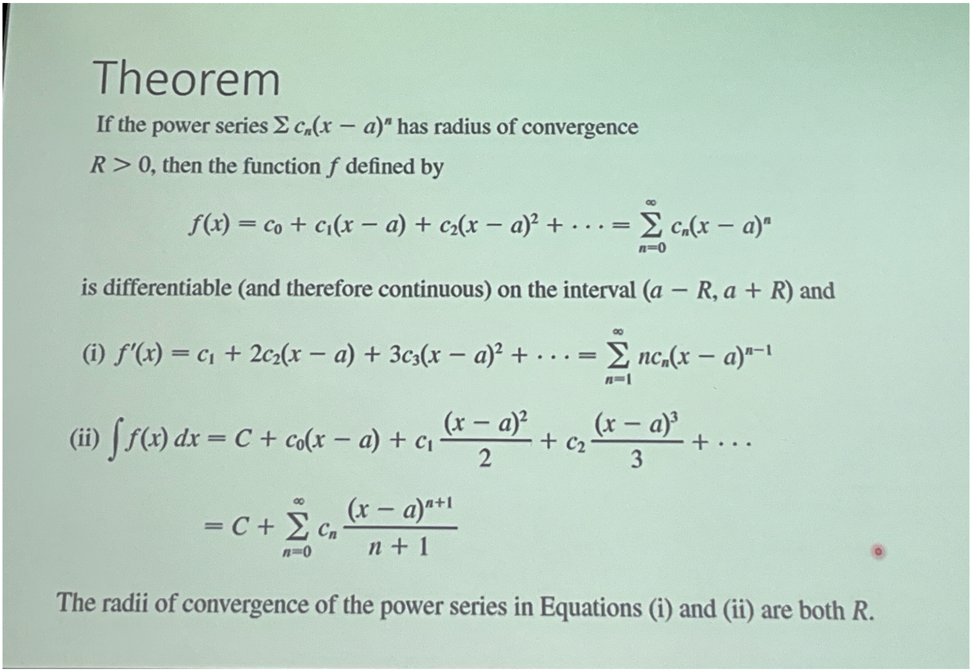
!! !

$$\frac{x^3}{x+2} \qquad x^3\left(\frac{1}{x+2}\right)$$

$\downarrow$

+ 3 [illegible] $x^n \to x^{n+3}$

$$x^3 \sum_{n=0}^{\infty} \frac{(-1)^n (x)^n}{2^{n+1}} = \frac{(-1)^n (x^{n+3})}{2^{n+1}}$$



Theorem

If the power series $\Sigma\, c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

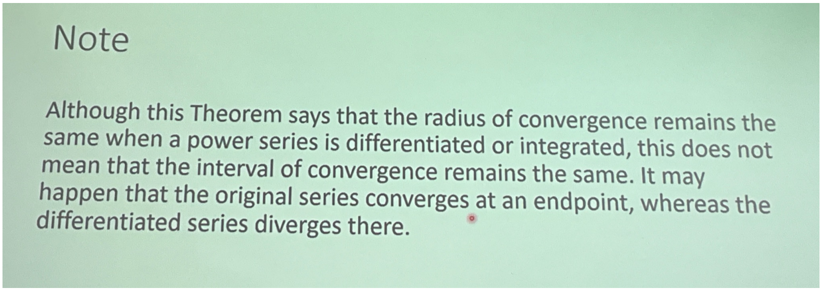
is differentiable (and therefore continuous) on the interval $(a - R, a + R)$ and

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

(ii) $\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$

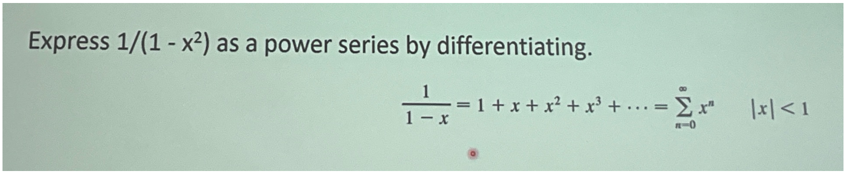
$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of the power series in Equations (i) and (ii) are both R.



Note

Although this Theorem says that the radius of convergence remains the same when a power series is differentiated or integrated, this does not mean that the interval of convergence remains the same. It may happen that the original series converges at an endpoint, whereas the differentiated series diverges there.

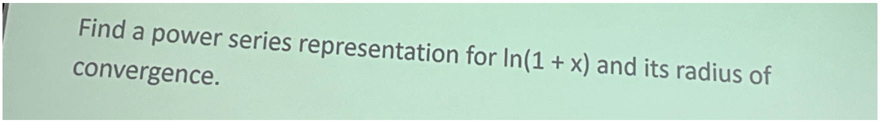


Express 1/(1 - x²) as a power series by differentiating.

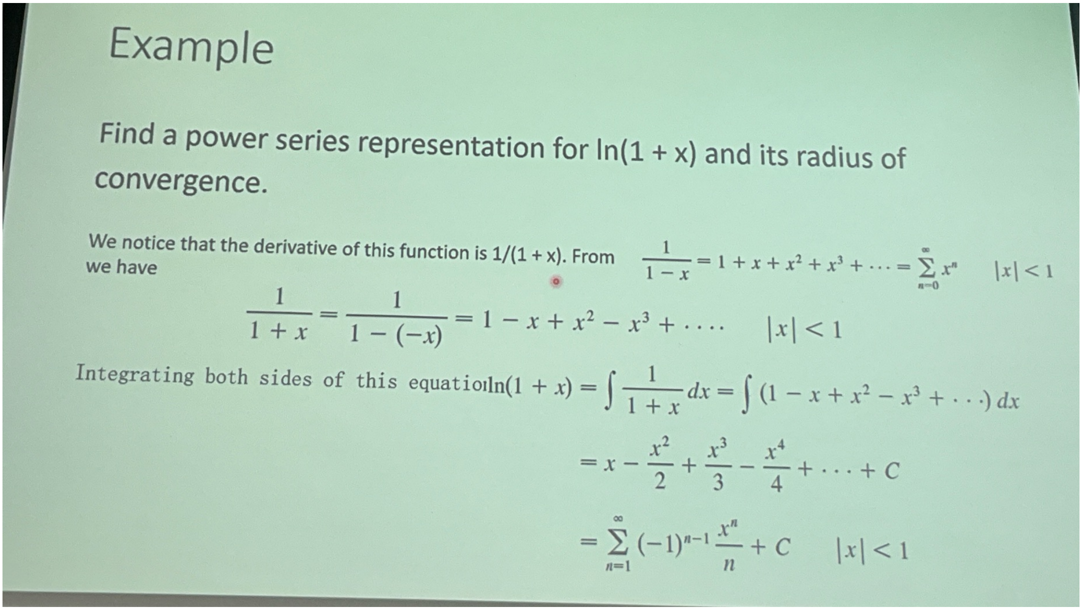
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$$

$$\frac{d}{dx}\left(\frac{1}{1-x^2}\right) = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots = \sum_{n=1}^{\infty} n x^{n-1}$$



Find a power series representation for ln(1 + x) and its radius of convergence.



Example

Find a power series representation for ln(1 + x) and its radius of convergence.

We notice that the derivative of this function is 1/(1 + x). From we have $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + \cdots \qquad |x| < 1$$

Integrating both sides of this equation $\ln(1+x) = \int \frac{1}{1+x}\,dx = \int (1 - x + x^2 - x^3 + \cdots)\,dx$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + C$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n} + C \qquad |x| < 1$$

Find a power series representation for f (x) = tan⁻¹x.

$$\frac{d}{dx} \tan^{-1} x = \int \frac{1}{1+x^2}$$

$$= \int 1 - x^2 + x^4 - x^6 + \dots \;) \, dx$$

$$= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

(a) Evaluate $\int \frac{1}{(1+x^7)}$ as a power series

$$= \int \frac{1}{(1-(-x^7))} = \quad a = 1, \quad r = (-x^7)$$

$$|-x^7| < 1$$

$$x^7 < 1$$

$$-1 < x < 1$$

$$= \sum_{n=0}^{\infty} (1)(-x^7)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n (x^{7n}) = 1 - x^7 + x^{14} - x^{21} + x^{28} - x^{35}$$

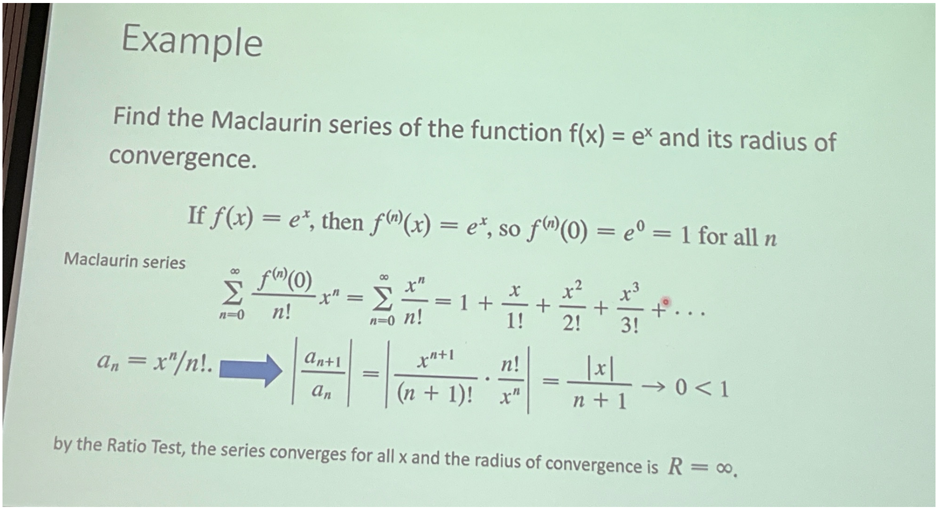
$$= \sum_{n=0}^{\infty} \int (-1)^n (x^{7n}) = C + x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \frac{x^{29}}{29} - \frac{x^{36}}{36}$$

$$7 \qquad 7 \qquad -7$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x^{7n+1})}{7n+1}$$

Come from integrated

Taylor Series and Maclaurin Series



Example

Find the Maclaurin series of the function f(x) = e^x and its radius of convergence.

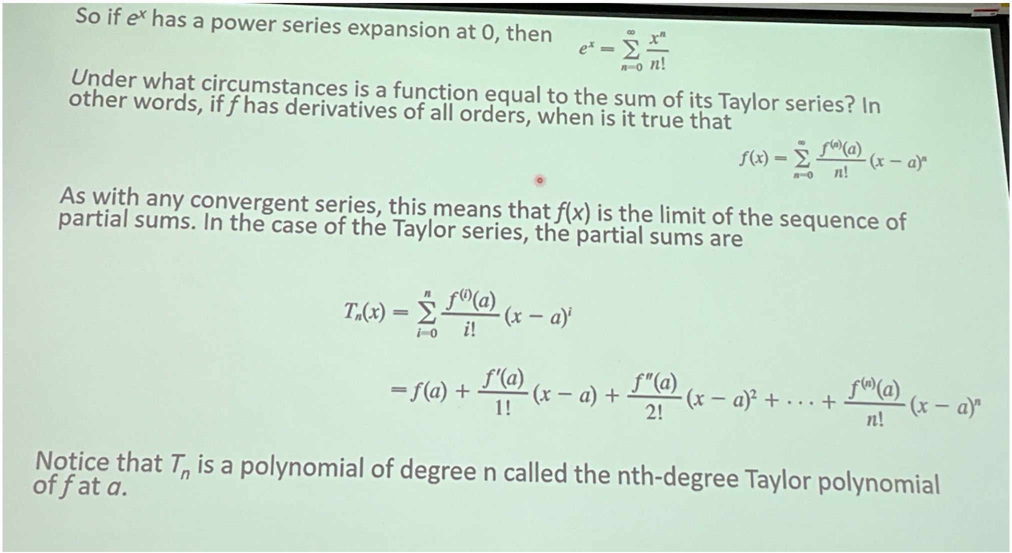
If $f(x) = e^x$, then $f^{(n)}(x) = e^x$, so $f^{(n)}(0) = e^0 = 1$ for all n

Maclaurin series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$a_n = x^n/n!. \Rightarrow \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \frac{|x|}{n+1} \to 0 < 1$$

by the Ratio Test, the series converges for all x and the radius of convergence is $R = \infty$.



So if e^x has a power series expansion at 0, then $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Under what circumstances is a function equal to the sum of its Taylor series? In other words, if *f* has derivatives of all orders, when is it true that

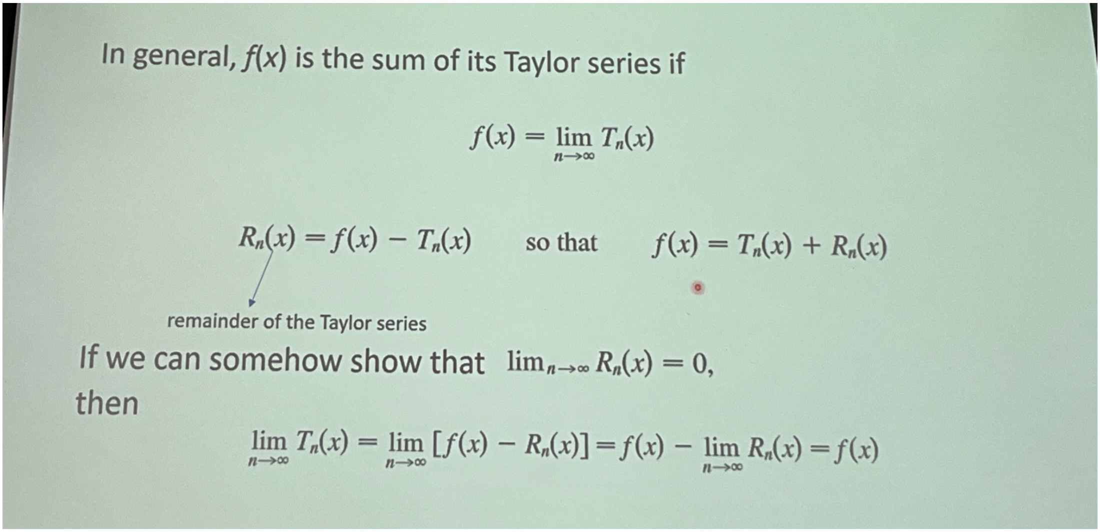
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

As with any convergent series, this means that *f*(*x*) is the limit of the sequence of partial sums. In the case of the Taylor series, the partial sums are

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!} (x-a)^i$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Notice that T_n is a polynomial of degree n called the nth-degree Taylor polynomial of *f* at *a*.



In general, $f(x)$ is the sum of its Taylor series if

$$f(x) = \lim_{n\to\infty} T_n(x)$$

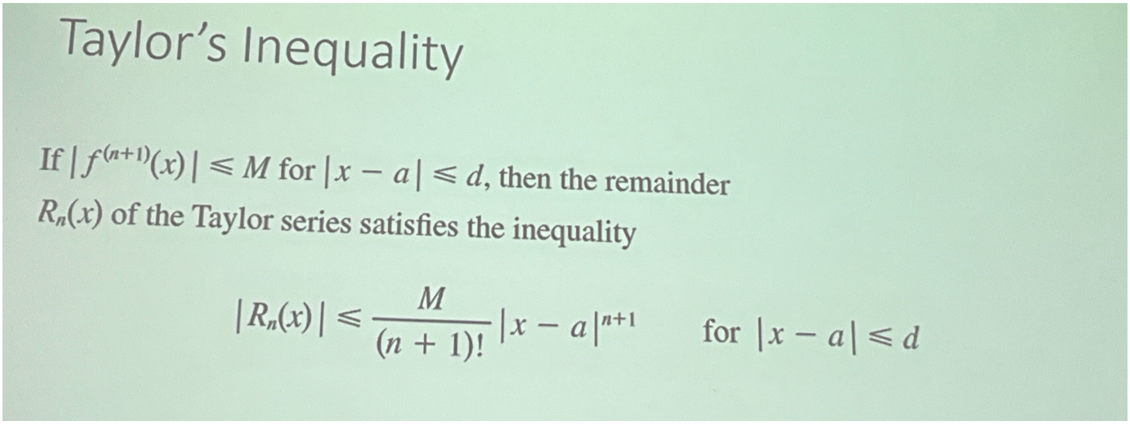
$$R_n(x) = f(x) - T_n(x) \qquad \text{so that} \qquad f(x) = T_n(x) + R_n(x)$$

remainder of the Taylor series

If we can somehow show that $\lim_{n\to\infty} R_n(x) = 0$,

then

$$\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} [f(x) - R_n(x)] = f(x) - \lim_{n\to\infty} R_n(x) = f(x)$$



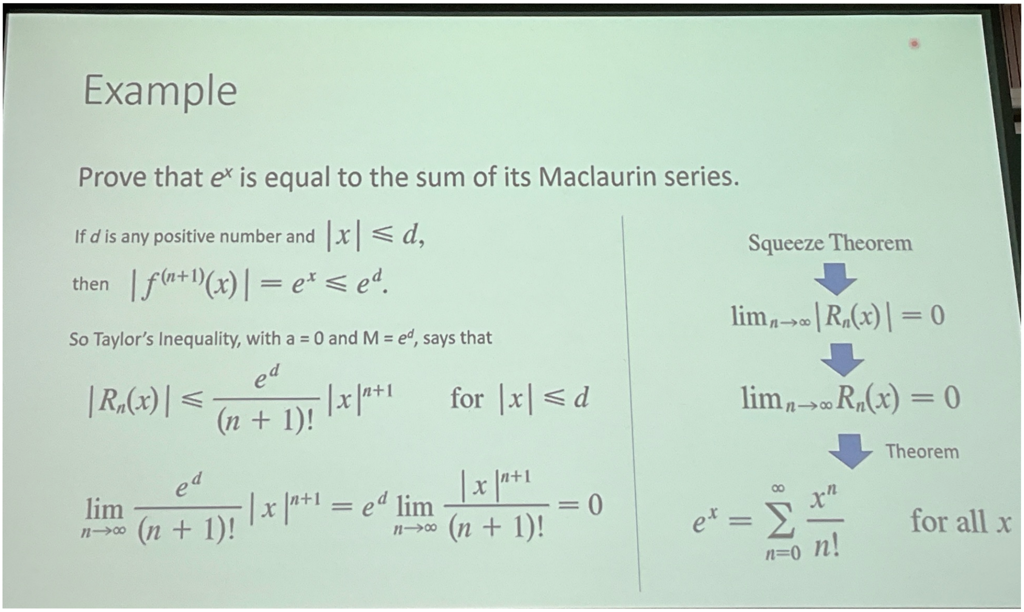
Taylor's Inequality

If $|f^{(n+1)}(x)| \leqslant M$ for $|x - a| \leqslant d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leqslant \frac{M}{(n+1)!}|x - a|^{n+1} \qquad \text{for } |x - a| \leqslant d$$

note: $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for every real number X.

Prove that e^x is equal to the sum of Maclaurin Series.



Example

Prove that e^x is equal to the sum of its Maclaurin series.

If d is any positive number and $|x| \leqslant d$,

then $|f^{(n+1)}(x)| = e^x \leqslant e^d$.

So Taylor's Inequality, with a = 0 and M = e^d, says that

$$|R_n(x)| \leqslant \frac{e^d}{(n+1)!}|x|^{n+1} \qquad \text{for } |x| \leqslant d$$

$$\lim_{n\to\infty} \frac{e^d}{(n+1)!}|x|^{n+1} = e^d \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

Squeeze Theorem

$$\lim_{n\to\infty} |R_n(x)| = 0$$

$$\lim_{n\to\infty} R_n(x) = 0$$

Theorem

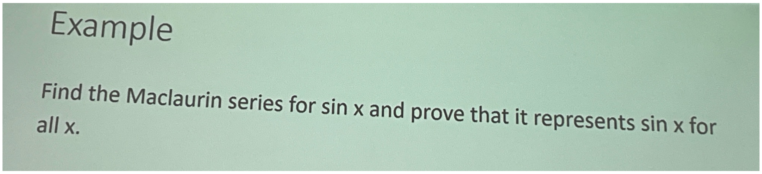
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{for all } x$$

Find the taylor Series for $f(x) = e^x$ at $a = 2$

$f^{(n)}(2) = e^2$ $\qquad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)} a}{n!} (x-a)^n$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$= \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-a)^n$$

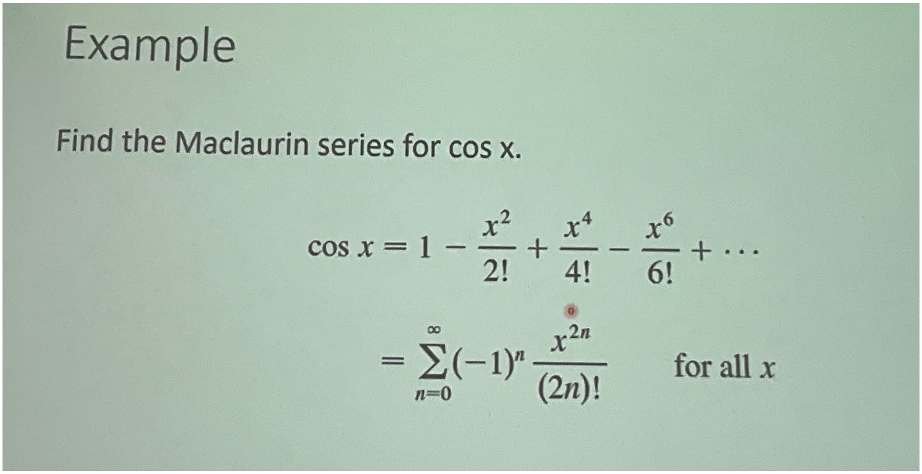


Example

Find the Maclaurin series for sin x and prove that it represents sin x for all x.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

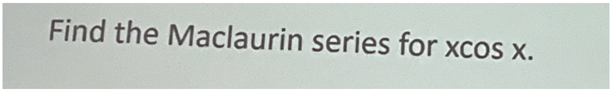


Example

Find the Maclaurin series for cos x.

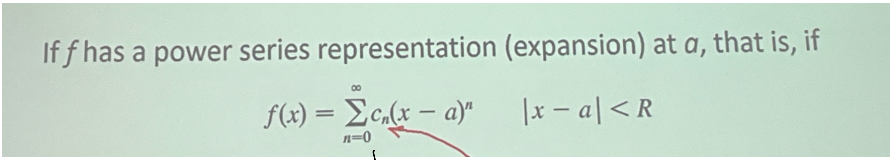
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad \text{for all } x$$



Find the Maclaurin series for xcos x.

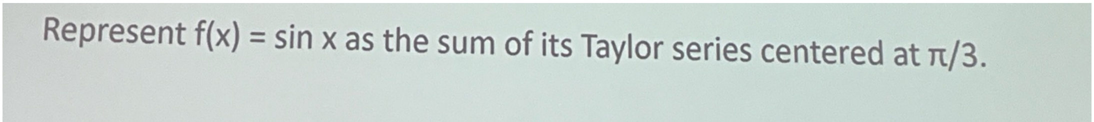
$$x\cos x = x\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n)!}$$



If *f* has a power series representation (expansion) at *a*, that is, if

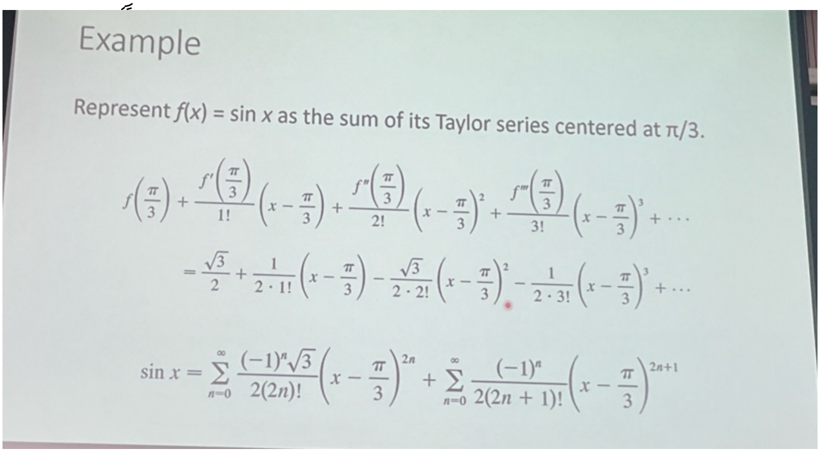
$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \qquad |x-a| < R$$

Taylor Series



Represent f(x) = sin x as the sum of its Taylor series centered at π/3.

$$= \sum_{n=0}^{\infty} \sin x\left(x-\frac{\pi}{3}\right)^n$$



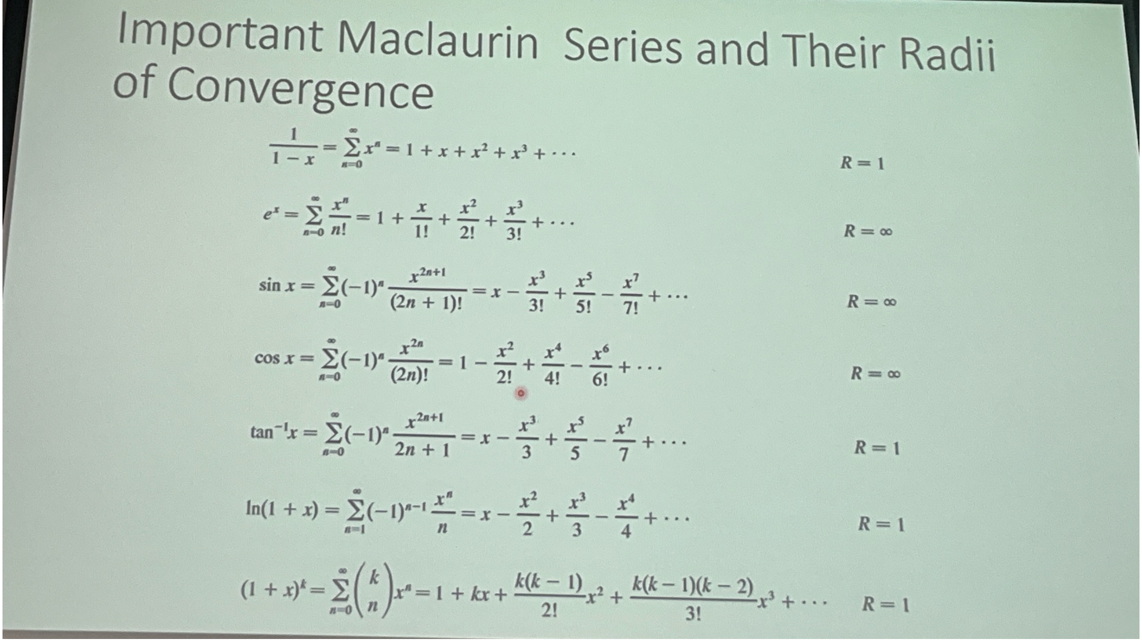
Example

Represent *f*(*x*) = sin *x* as the sum of its Taylor series centered at π/3.

$$f\left(\frac{\pi}{3}\right) + \frac{f'\left(\frac{\pi}{3}\right)}{1!}\left(x-\frac{\pi}{3}\right) + \frac{f''\left(\frac{\pi}{3}\right)}{2!}\left(x-\frac{\pi}{3}\right)^2 + \frac{f'''\left(\frac{\pi}{3}\right)}{3!}\left(x-\frac{\pi}{3}\right)^3 + \cdots$$

$$= \frac{\sqrt{3}}{2} + \frac{1}{2\cdot 1!}\left(x-\frac{\pi}{3}\right) - \frac{\sqrt{3}}{2\cdot 2!}\left(x-\frac{\pi}{3}\right)^2 - \frac{1}{2\cdot 3!}\left(x-\frac{\pi}{3}\right)^3 + \cdots$$

$$\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n\sqrt{3}}{2(2n)!}\left(x-\frac{\pi}{3}\right)^{2n} + \sum_{n=0}^{\infty}\frac{(-1)^n}{2(2n+1)!}\left(x-\frac{\pi}{3}\right)^{2n+1}$$



Important Maclaurin Series and Their Radii of Convergence

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1}x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad R = 1$$

$$(1+x)^k=\sum_{n=0}^{\infty}\binom{k}{n}x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

!

$$= 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}$$

5 7 9

$a = 1$ $r =$