Logic

Logical operators: $\neg$ (not or negations), $\wedge$ (and), $\vee$ (or), $\Rightarrow$ (implication)

$\Leftrightarrow$ (equivalence), | (sheffer stroke), $\downarrow$ Pierce Arrow

Tautology: Always true (compound proposition)

Contradiction: Always false (compound proposition)

Contingency: neither tautology / contradiction

$| : \neg(A \wedge B)$

Logically equivalent?

Compound state A and B are logically equivalent if $A \Leftrightarrow B$ is a tautology

* Notation $A \equiv B$ denotes A and B are logically equivalent.

Exercise 1.2

Let A and B be statements. Use a truth table to show that

$$\neg(A \Rightarrow B) \quad \equiv \quad A \wedge \neg B.$$

(2 Marks)

A	B	¬A	¬B	A => B	¬(A=>B)	A ∧ ¬B	¬(A=>B) <=> A ∧ ¬B
T	T	F	F	T	F	F	T
F	T	T	F	T	F	F	T
T	F	F	T	F	T	T	T
F	F	T	T	T	F	F	T

From the table column, we can see that ¬(A=>B) is equal to A∧¬B that all values is true, in other words they are a tautology.

Exercise 1.3

Explain in your own words the difference between the statements

$$\exists_{0\in\mathbb{Q}}\ \forall_{a\in\mathbb{Q}}\ a+0=0+a=a \qquad \text{and} \qquad \forall_{a\in\mathbb{Q}}\ \exists_{0\in\mathbb{Q}}\ a+0=0+a=a.$$

(4 Marks)

There is exist 0 in elements of rationals number that for all any element "a" in the set of rational numbers will lead to a+0 = 0+a = a

For all a in set of rational number there will exist a 0 in in Q such that a+0 = 0+a = a.

Both of this is true, but by having universal quantifier in front of exisensial quantifier and vice versa, will lead to different ways of thinking (logic, Ex: P(x, y))

let take x: football team y: players who play football

$\exists_x \forall_y$ → There is football team that have all players who play football

$\forall_y \exists_x$ → for all football players there will be some players who in the football team.

Set and logic

Disjunction = Or

Conjunction = and

negations = $\neg$ or opposite values

Disjunctive normal form = use of $\neg, \land, \lor$ ex $(\neg A \land B) \lor (\neg A \land \neg B)$

Demorgan law = $\neg(p \land q) = \neg p \lor \neg q$
$\neg(p \lor q) = \neg p \land \neg q$

use $\neg, \land$

Scheffer stroke : $A \mid B :\equiv \neg(A \land B)$

Induction

$f_n + f_{n+1} = f_{(n+2)}$

Rule

(i) $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$

(ii) $a, b > 0$ imply $\sqrt{ab} \leq \frac{a+b}{2}$

AMGM : $\frac{x_1 + x_2 + \dots + x_n}{n} \geq \sqrt[n]{x_1 \cdot x_2 \cdot x_3 \cdots x_n}$

$$\frac{a_1 + a_2 + \dots + a_n}{n} \geq \sqrt[n]{a_1 \cdot a_2 \cdot a_3 \cdots a_n}$$

Sets

Set: An unordered collection of object.

operations: $\cup$ (union), $\cap$ (intersect), $\setminus$ (difference), A^c (complement of set A)

Natural number properties (1, 2, 3, 4 . . . .)

	Addition	Multiplication
Associativity	$a+(b+c)=(a+b)+c$	$a\cdot(b\cdot c)=(a\cdot b)\cdot c$
Existence	$a+0=0+a=a$	$a\cdot 1=1\cdot a=a$
Commutativity	$a+b=b+a$	$a\cdot b=b\cdot a$
Distributivity	$a\cdot(b+c)=a\cdot b+a\cdot c$	

Rational number

→ The ratio of two integers, a and b, where b is not equal to zero,

$$\frac{a}{b}$$

a and b are integers and $b \neq$

same as natural number

Add Inverse element

$(-a)+a=a+(-a)=0$

Complex number

note $1+i=\sqrt{2}\cdot e^{i\pi/4}$ $\quad 1-i=\sqrt{2}\cdot e^{\frac{-i\pi}{4}}$

$e^{\frac{i\pi}{2}}=i$

$\sqrt{2}$

500
$9\sqrt{2002}$
20
00

$$|z|=\sqrt{\bar{z}\cdot z}$$

$$|z|^2=\bar{z}\cdot z$$

$$\overline{z_2}\,z_1+\bar{z}_2 z_1=2\,\mathrm{Re}(\bar{z}_2 z_1)$$

* If a sequence have limit, it must be bounded.

$$\mathrm{Re}_{z_1}=\frac{z_1+\bar{z}_1}{2}$$

$$(\mathrm{Im}\,z_1)i=\frac{z_1-\bar{z}_1}{2}$$

$$\overline{z_1+z_2}=\bar{z}_1+\bar{z}_2$$

$$\overline{z_1-z_2}=\bar{z}_1-\bar{z}_2$$

$$\overline{z_1 z_2}=\bar{z}_1\cdot\bar{z}_2$$

$$\left(\frac{\overline{z_1}}{z_2}\right)=\frac{\overline{z_1}}{z_2},\ z_2\neq 0$$

Polar form: $(|z|, \arg z)$ or (r, θ)

1. $r=\sqrt{a^2+b^2}$

2. $a=r\cos(\theta)$, $b=r\sin(\theta)$

Euler's Formula : $e^{i\theta} = \cos\theta + i\sin\theta$

De Moivre's Theorem

$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$

complex number in exponential form : $z = r \cdot e^{i\theta}$

1. $z_1 \cdot z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$ 2. $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$

Sequence and Subsequence

$\{a_n\} \leq \{a_{n+1}\}$: Monotonically increasing

$\{a_n\} \geq \{a_{n+1}\}$: monotonically decreasing

$\{a_n\} < \{a_{n+1}\}$: strictly increase , monotonic

$\{a_n\} > \{a_{n+1}\}$: Monotonic, strictly decreasing

Sub sequence : Let $\{a_n\}$ be a sequence. If n_1, n_2 , are positive integers such that $n_k < n_{k+1}$ for each $k \in \mathbb{N}$, then $\{a_{n_k}\}$ whose terms are $a_{n_1}, a_{n_2}, \ldots$ is called a subsequence of $\{a_n\}$

Convergence and Divergence of Sequence

1. $\lim_{n\to\infty} a_n = L$

   $\underset{\varepsilon>0}{\forall}$ $\underset{N\in\mathbb{N}}{\exists}$ $\underset{n>N}{\forall}$ then $|a_n - L| < \varepsilon$

2. $\lim_{n\to\infty} a_n = \infty$

   $\underset{M>0}{\forall}$ $\underset{N\in\mathbb{N}}{\exists}$ $\underset{n>N}{\forall}$ then $|a_n| > M$

Important Results / Theorem

but bounded sequence not always convergent.

A convergent sequence is bounded, A convergent sequence has precisely a unique limit

- let $\{a_n\}$ be a sequence. $\lim_{n\to\infty} a_{2n} = a$ and $\lim_{n\to\infty} a_{2n+1} = a$ then $\lim_{n\to\infty} a_n = a$

limit law

Sum / difference law

$$\lim_{n\to\infty} (a \pm b) = \lim_{n\to\infty} a \pm \lim_{n\to\infty} b$$

Constant multiple law

$$\lim_{n\to\infty} c \cdot a = c \cdot \lim_{n\to\infty} a$$

Product law

$$\lim_{n\to\infty} (a \cdot b) = \lim_{n\to\infty} a \cdot \lim_{n\to\infty} b$$

quotient law

$$\lim_{n\to\infty} \frac{a}{b} = \frac{\lim_{n\to\infty} a}{\lim_{n\to\infty} b} \rightarrow \text{if } \lim_{n\to\infty} b_n \neq 0$$

Power law

$$\lim_{n\to\infty} a_n^p = \left(\lim_{n\to\infty} a_n\right)^p \quad \text{if } p>0,\ a_n \geq 0$$

$$\lim_{n\to\infty} \sqrt[n]{a} = \lim_{n\to\infty} a^{\frac{1}{n}}$$

Squeeze Theorem

$$a_n \le b_n \le c_n$$

that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = a$ Then $\lim_{n\to\infty} b_n = a$

Ex. $\lim_{n\to\infty} \left(\frac{1}{4n^2+1} + \frac{2}{4n^2+2} + \cdots + \frac{n}{4n^2+n} \right)$ $\qquad 1+2+\cdots+n = \frac{n(n+1)}{2}$

$$b_n = \frac{1}{4n^2+1} + \frac{2}{4n^2+2} + \cdots + \frac{n}{4n^2+n}$$

$$\frac{1+2+\cdots+n}{4n^2+n} < b_n < \frac{1+2+\cdots+n}{4n^2+1}$$

$$\frac{\frac{n(n+1)}{\cancel{2}}}{4n^2+1}$$

$\lim = \frac{1}{8}$

$\lim = \frac{1}{8}$

accumulation point

A point which value of the set come arbitrarily close.

limit: number for any given distance, all terms of the sequence eventually are within that distance of limit.

Accumulation point: for any distance, there will be other succeeding sequence term within that distance of the accumulation point.

- Any limit is also an accumulation point.
- A sequence can have at most a single limit, but can have zero or more accumulation points.

Subsequence

✱ if converge, each subsequence limit must be the same
else: it diverge.

Show $\underset{a_n}{=} (-1)^n \cdot \frac{n+1}{n}$ is divergent. (Hint: check whether its subsequence converges to same limit)

✱ $a_n = \{1, 2, 3, 4 \ldots\}$

$a_{2n} = \{2, 4, 6, \ldots\ldots\}$

$$a_{2n} = \underbrace{(-1)^{2n}}_{=1} \cdot \frac{2n+1}{2n} = \frac{2n+1}{2n} = 1 + \frac{1}{2n}$$

$$\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} 1 + \frac{1}{2n} = \lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{2n} \quad \text{(sum law)}$$

$$= 1 + 0 = 1$$

$$a_{2n+1} = \underbrace{(-1)^{2n+1}}_{-1} \cdot \frac{(2n+1)+1}{2n+1} = -1 \cdot \frac{(2n+2)}{2n+1} = -1 \cdot \left(1 + \frac{1}{2n+1}\right) = -1 - \frac{1}{2n+1}$$

$$\lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty} -1 - \lim_{n\to\infty} \frac{1}{2n+1} = -1 - 0 = -1 \quad \text{(difference law)}$$

Since the subsequences do not converge to same limit, then a_n is divergent.

Prove $\lim_{n\to\infty} \frac{3n+1}{2n+1} = \frac{3}{2}$ $\qquad \lim_{n\to\infty} a_n = L$

let $\varepsilon > 0$ $\qquad \underset{\varepsilon>0}{\forall} \quad \underset{N \in \mathbb{N}}{\exists} \quad \underset{n>N}{\forall} \quad |a_n - L| < \varepsilon$

$$|a_n - L| < \varepsilon = \left| \frac{3n+1}{2n+1} - \frac{3}{2} \right| < \varepsilon$$

$$= \left| \frac{6n+2-(6n+3)}{(2n+1)(2)} \right| = \frac{1}{4n+2} < \frac{1}{4n}$$

since we want to prove $\left| \frac{3n+1}{2n+1} - \frac{3}{2} \right| < \varepsilon$, we only need.

$\frac{1}{4n} < \varepsilon \Rightarrow n > \frac{1}{4\varepsilon}$, $\forall \varepsilon > 0$, choose $N = \left[\frac{1}{4\varepsilon}\right]$: when $n > N$

we got $\left| \frac{3n+1}{2n+1} - \frac{3}{2} \right| < \frac{1}{4n} < \frac{1}{4n} = \frac{1}{4\left(\frac{1}{4\varepsilon}\right)} = \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$ $\quad$ Thus $\lim \frac{3n+1}{2n+1}$ proves

Consider $a_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$, $\forall n \in N$, show a_n is bounded and monotone, then deduce a_n is converge

Prove monotonic $a_{n+1} - a_n = \left\{\cancel{1} + \cancel{\frac{1}{2^2}} + \cancel{\frac{1}{3^2}} + \cdots + \cancel{\frac{1}{n^2}} + \frac{1}{(n+1)^2}\right\} - \left\{\cancel{1} + \cancel{\frac{1}{2^2}} \cdots + \cancel{\frac{1}{n^2}}\right\}$

$= \frac{1}{(n+1)^2} > 0$ strictly increase for $n \in N$

as $\frac{1}{n^2} < \frac{1}{n^2 - n} < \frac{1}{n(n-1)} = \frac{1}{n} - \frac{1}{(n-1)}$ Thus it is monotonic

Bounded $a_n = \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}\right) < a_n \left(1 + 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n-1} - \frac{1}{n}\right.$

$a_n < 2 - \frac{1}{n}$

$\lim_{n \to \infty} 2 - \lim_{n \to \infty} \frac{1}{n}$

$= 2$

Thus a_n is bounded above by 2

SERies

$a_1 + a_2 + a_3 \dots$ is a Series

Written:

$$\sum_{n=1}^{\infty} a_n$$

OR

$$\sum_{k=1}^{n} a_k$$ For A Partial Sum

"n"s

$$\sum_{k=1}^{n} a_k = S_n$$

S_n Must Exist

* For the next "n" There is a new S_n"

This creates A seq. of partial sums.

Telescoping Series

Ex.

$$\sum_{n=1}^{\infty} \frac{4}{4n^2-1}$$

$$= \frac{2}{(2n-1)} - \frac{2}{(2n+1)}$$

$$\frac{T}{4n^2-1} \to \left[\frac{4}{\cancel{(2n-1)}\cancel{(2n+1)}} = \frac{A}{\cancel{(2n-1)}} + \frac{\cancel{B}}{\cancel{(2n+1)}} \right] \cancel{(2n-1)}\cancel{(2n+1)}$$

$$= (2n+1)A + B(2n-1)$$

$4 = 2A +$ $\quad A = 2$

$n = \frac{1}{2}$

$n = -\frac{1}{2}$

$-2B = -4 \quad b = -2$

$$\sum_{n=1}^{\infty} \frac{4}{4n^2-1} = \sum_{n=1}^{\infty} \left(\frac{2}{2n-1} - \frac{2}{2n+1} \right)$$

$$S_n = \left(\frac{2}{1} - \cancel{\frac{2}{3}}\right) + \left(\cancel{\frac{2}{3}} - \cancel{\frac{2}{5}}\right) + \left(\cancel{\frac{2}{5}} - \cancel{\frac{2}{7}}\right) \cdots$$

$$\left(\frac{2}{2n-1} - \cancel{\frac{2}{2n+1}}\right)$$

$$S_n = 2 - \frac{2}{2n-1}$$

$$\lim_{n \to \infty} 2 - \frac{2}{2n-1}$$

$$= 2$$

A Geometric series

$$\sum_{n=1}^{\infty} a r^{n-1} = a + ar + ar^2 + ar^3 + \cdots$$

or

$$\sum_{n=0}^{\infty} a r^n$$

☆ Converge when $|r| < 1$ or $-1 < r < 1$

→ Diverge $|r| \geq 1$

☆ The Sum is

$$\sum_{n=1}^{\infty} a r^{n-1} = \frac{a}{1-r}$$

$$\sum_{n=1}^{\infty} \frac{2^{n-1}}{5^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{25} \cdot \left(\frac{2}{5}\right)^{n-1}$$

$$\frac{2^{n-1}}{(5)(5)^n} \qquad \frac{1}{5^n} \cdot \left(\frac{2}{5}\right)^{n-1}$$

Ex $\sum_{n=1}^{\infty} 3\left(-\frac{1}{2}\right)^{n-1} \longrightarrow a = 3, \quad r = -\frac{1}{2}$

$|r| < 1 \longrightarrow$ series converges and sum is $= \frac{3}{1+\frac{1}{2}} = 2$

Ex $\sum_{n=1}^{\infty} a_n = \left(\frac{5}{3} - \frac{5}{9} + \frac{5}{27} - \frac{5}{81} + \cdots\right)$

$= \frac{5}{3}\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} \cdots\right)$

$= \frac{5}{3}\left(\frac{1}{3^0} - \frac{1}{3^1} + \frac{1}{3^2} - \frac{1}{3^3} \cdots\right)$

$= \frac{5}{3}\left(-\frac{1}{3}\right)^{n-1}$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{5}{3}\left(-\frac{1}{3}\right)^{n-1}$$

$a = \frac{5}{3}, \quad r = -\frac{1}{3}$ so converges

as $|r| < 1$

$\text{sum} = \frac{\frac{5}{3}}{1-\left(\frac{1}{3}\right)} =$

Harmonic Sereis

If a series $\sum_{n=1}^{\infty} a_n$ converges, $\lim_{n \to \infty} a_n = 0$

If $\lim_{n \to \infty} a_n \neq 0$ or D.N.E

The series $\sum_{n=1}^{\infty} a_n$ Diverge.

Ratio test

1) $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ is absolutely convergent

2) $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ or diverges $\Rightarrow \sum_{n=1}^{\infty} a_n$ is divergent

3.) $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1 \Rightarrow$ Inconclusive

Ex 1 $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} \right|$$

$$= \lim_{n \to \infty} \left| \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \right| = \lim_{n \to \infty} \left| \frac{2^{n+1}}{2^n} \cdot \frac{n!}{(n+1)!} \right|$$

$$\lim_{n \to \infty} \left| 2 \cdot \frac{\cancel{n(n-1)(n-2) \cdots 2 \cdot 1}}{(n+1) \cdot \cancel{n(n-1)(n-2) \cdots 2 \cdot 1}} \right|$$

$$= \lim_{n \to \infty} \left| 2 \cdot \frac{1}{(n+1)} \right|$$

$$= \lim_{n \to \infty} \left| \frac{2}{n+1} \right|$$

$$\lim_{n \to \infty} = 0 < 1$$ Converge

Root test

1) $\lim_{n \to \infty} \sqrt[n]{a_n} = L < 1 \to$ converges

2) $\lim_{n \to \infty} \sqrt[n]{a_n} = L > 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ is diverge

3) $\lim_{n \to \infty} \sqrt[n]{a_n} = 1$ inconclusive

Ex 4 $\quad \sum_{n=1}^{\infty} \left(\frac{n+1}{2n-1}\right)^n$

$$\lim_{n \to \infty} \sqrt[n]{\left(\frac{n+1}{2n-1}\right)^n} = \lim_{n \to \infty} \frac{n+1}{2n-1} = \frac{1}{2} < n$$

Excercise 1

$A \Longleftrightarrow B \equiv (A \wedge B) \vee (\neg A \wedge \neg B)$

A	B	¬A	¬B	A∧B	¬A∧¬B	(A∧B)∨(¬A∧¬B)	A <=> B
T	T	F	F	T	F	T	T
T	F	F	T	F	F	F	F
F	T	T	F	F	F	F	F
F	F	T	T	F	T	T	T

Tautology so it is logically equivalent

Exercise 3

show $(1+x_1)(1+x_2)\ldots(1+x_n) \geq 1+x_1+x_2$

check at $n=1$ $\qquad 1+x_1 = 1+x_1$ $A(1)$ is true

Assume n is true such that $n+1$ is hold true

Exercise 4

find $\lim_{n\to\infty} \left(\frac{1^2}{n^3+2} + \frac{2^2}{n^3+2} + \frac{3^2}{n^3+2} + \ldots + \frac{n^2}{n^3+2}\right.$

$1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6} \cdot \frac{1}{n^3+2} = \frac{2n^3+3n^2+n}{6n^3+12} =$

$$\frac{\lim_{n\to\infty}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)}{\lim_{n\to\infty}\left(6+\frac{12}{n^3}\right)} = \text{quotient law}$$

Exercise 5

Consider $a_n = \frac{2n-3}{3n+4}$, $\forall n \in \mathbb{N}$ show a_n is bounded

and monotone, then deduce a_n is convergent

$$a_{n+1} - a_n = \frac{2(n+1)-3}{3(n+1)+4} - \frac{2n-3}{3n+4}$$

$$= \frac{2n+2-3}{3n+7} - \frac{2n-3}{3n+4}$$

$$= \frac{2n-1}{3n+7} - \frac{2n-3}{3n+4} = \frac{10n+7}{(3n+7)(3n+4)} > 0 \quad \text{monotonically increases}$$

$$a_n < \frac{2n-3}{3n} < \frac{2n-3}{3n} < \frac{2}{3} \quad \text{and} \quad a_1 = -\frac{1}{7}$$

$$-\frac{1}{7} \le a_n \le \frac{2}{3} \quad \text{monotone and bounded so it's convergent}$$

Monotonic sequence theorem

$a(n)$ is increasing

$a_n \le a_{n+1}$ for all n

$\frac{a_{n+1}}{a_n} \ge 1$ for all n, $a_n > 0$

$a_{n+1} - a_n \ge 0$ for all n

$a(n)$ is decreasing

$a(n) \ge a_{n+1}$ for all n

$\frac{a_{n+1}}{a_n} \le 1$ for all n, $a_n > 0$

$a_{n+1} - a_n \le 0$ for all n

Proove $\lim\limits_{n \to \infty} \dfrac{\sqrt{n^2+a^2}}{n} = 1$

let $\varepsilon > 0$ such that $\underset{\varepsilon>0}{\forall}\ \underset{n\in N}{\exists}\ \underset{n>N}{\forall}\ |a_n - L| < \varepsilon$

$$\left| \frac{\sqrt{n^2+a^2}}{2} - 1 \right| = \frac{\sqrt{n^2+a^2} - n}{n} = \frac{a^2}{n(\sqrt{n^2+a^2}+n)} < \frac{a^2}{2n^2}$$

Suppose X_n is bounded, and $\lim\limits_{n \to \infty} Y_n = 0$. Prove $\lim\limits_{n\to\infty} X_n \cdot Y_n = 0$

Since X_n is bounded $\underset{M>0}{\exists}\ \underset{n\in N}{\forall}$, $|X_n| \le M$

$\underset{\varepsilon>0}{\forall}$, since $\lim\limits_{n\to\infty} Y_n = 0$, so for $\varepsilon_1 = \dfrac{\varepsilon}{M} > 0$, $\exists\, N \in N$ such that

when $n > N$, we have $|Y_n - 0| = |Y_n| < \varepsilon_1 = \dfrac{\varepsilon}{M}$

Then, $|X_n Y_n - 0| = |X_n||Y_n| < M \cdot \dfrac{\varepsilon}{M} = \varepsilon$.