1.4 Continuity

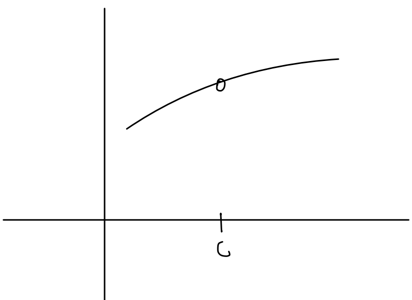
A curve is continuous If it has no "Holes", "Breaks", OR Asymptotes

Continuous At A point "C" If

1. $f(c)$ Is defined.

2. $\lim_{x \to c} f(x)$ Must Exist

3. $\lim_{x \to c} f(x) = f(c)$



Not Cont. At "C".

Definition of Continuous

f continuos at x_0 $\iff$ $\forall_{\varepsilon > 0} \ \exists_{\delta > 0} \ \forall_{x \in \text{Dom } f}$ $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

$\iff f(x_0) = \lim_{x \to x_0} f(x)$

Theorem: f continuous at x_0 $\iff$ $\forall_{(a_n)}$ $a_n \to x_0 \Rightarrow f(a_n) \to f(x_0)$

Ex. Are these continuous At $x = 2$?

$f(x) = \dfrac{x^2 - 4}{x - 2}$; $g(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ 3, & x = 2 \end{cases}$; $h(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$

no as

$f(x)$ is not defined

NO

$\lim_{x \to c} f(x) \neq c$

If f is continuous At every point between a & b, Then f is cont on (a, b)

<u>Prove</u> $f(x) = \sqrt{16 - x^2}$ Is continuous on $[-4, 4]$

1. Check $(-4, 4)$

Prove $\lim_{x \to c} f(x) = f(c)$

$$\lim_{x \to c} f(x) = \lim_{x \to c} \sqrt{16 - x^2} = \sqrt{16 - c^2} = f(c)$$

$$\lim_{x \to -4^+} \sqrt{16 - x^2} = 0 \qquad \lim_{x \to 4^-} \sqrt{16 - x2} = 0$$

<u>Properties</u> f and g are continuous at Point c.

1. $f + g$
2. $f - g$
3. $f \cdot g$

$\left.\begin{matrix} \\ \\ \end{matrix}\right\}$ Are cont. At Point c.

4. $\frac{f}{g}$ Is cont at "c" unless. $g(c) = 0$

If $g(c) = 0$, There is

A <u>Discontinuity</u> At "c"

for $\frac{f}{g}$

If A Funct is cont At C;

Note $f(x) = |x|$ cont everywhere

$$\lim_{x \to c} f(x) = f(c) \qquad \lim_{x \to c} P(x) = P(c)$$

<u>Compositions</u>: If $\lim_{x \to c} g(x) = L$ and f is

continuous At L

Then $\underbrace{\lim_{x \to c} f[g(x)] = f(L) = f\left[\lim_{x \to c} [g(x)]\right)}_{\text{We can separate limit by composition.}}$

<u>Ex</u> $\lim_{x \to 4} |10 - 3x^2| \longrightarrow \left|\lim_{x \to 4} 10 - 3x^2\right|$

$|-38| = 38$

<u>IVT</u>

F is cont. on $[a, b]$, say K is between $f(a) \neq f(b)$ then there exist

$c \in (a, b)$ where c is between a and b, and $f(c) = K$.

* useful in root problem.

Def of Limit Function

If $I \supset (a, \infty)$ for some $a \in \mathbb{R}$, then we define.

$$\lim_{x\to\infty} f(x) = L \iff \forall_{\varepsilon>0} \ \exists_{c>0} \ \forall_{x>c} \ |f(x)-L| < \varepsilon$$

"x large - close to ∞"

$$\iff \forall_{\varepsilon>0} \ \exists_{c>0} \ \forall_{x\in I} \ x>c \Rightarrow |f(x)-L| < \varepsilon$$

$$\lim_{x\to x_0} f(x) \quad f(x) = L \iff \forall_{\varepsilon>0} \ \exists_{\delta>0} \ \forall_{x\in I\setminus\{x_0\}} \ |x-x_0| < \delta \Rightarrow |f(x)-L| < \varepsilon$$

"x close to x_0"

2. Completing the square

$2x^2+3x-4$

$a=2 \quad B=3, \ C=-4$

$\gamma = C - \alpha\beta^2$

$\alpha = 2, \ \beta = \frac{-3}{4}, \ \gamma = -4-(2)(\frac{-3}{4})^2$

$= -\frac{41}{8}$

$\Rightarrow 2x^2+3x-4 = 2(x+\frac{3}{4})^2 - \frac{41}{8}$

$P(x) = 2(x+\frac{3}{4})^2 - \frac{41}{8}$ will be minimal when the square term vanishes,

when $x = -\frac{3}{4}$

Big O and Small O

Definition $\exists\, c>0$ $\exists\, a>0$ $\forall\, x \in I$ as $x \to \infty$

as $x > a \Rightarrow |f(x)| < c\,|(g(x))|$ $\Leftrightarrow |x - x_0| < \delta$

For Big O = f(x) is not significanty greater or equal to g(x) in a neighborhood of x_0

For Small O = f(x) is always less than g(x) near a neighbourhood of x_a

Theorem

Big O : for $x_0 \in \infty, R$

prove by $\lim\limits_{x \to \infty, x_a} \left| \dfrac{f(x)}{g(x)} \right|$ $c < x < \infty$, or for $x_0 \in R$ $\left| \dfrac{f(x)}{g(x)} \right| = L$ and $L \geq 0$

Small o : for $x_0 \in \infty, R$

Prove by $\lim\limits_{x \to \infty, x_0} \left| \dfrac{f(x)}{g(x)} \right| = 0$

Polynomial Division

ex. $p(x) = x^2 + 7x + 12$, $g(x) = x + 3$

$$
\begin{array}{r|l}
 & x + 4 \\
\hline
x+3 & x^2 + 7x + 12 \\
- & x^2 + 3x \\
 & \quad 4x + 12 \\
- & \quad 4x + 12 \\
\hline
 & \quad\ 0 \quad 0
\end{array}
$$

Thus root $= (x+3)(x+4)$ #

Factorization

ex. $6x^4 - 30x^2 + 24 \rightarrow$ set $y = x^2$

$\dfrac{6y^2 - 30y + 24}{6} = 0$

$y^2 - 5y + 4 = 0$

$(y-4)(y-1) = 0$

factor $= (x^2 - 4)(x^2 - 1)$

Factorize ex. 2

Find for $2x^5 + 3x^4 - 4x^3 - 3x^2 + 2x$

first try some root $x = -1$ $\rightarrow$ one factor $= (x+1)$

$$
\begin{array}{r|l}
 & 2x^4 + x^3 - 5x^2 + 2x \\
\hline
= x+1 & 2x^5 + 3x^4 - 4x^3 - 3x^2 + 2x \\
 & 2x^5 + 2x^4 \\
 & \quad x^4 - 4x^3 \\
 & \quad \cancel{x}^4 + x^3 \\
 & \qquad -5x^3 - 3x^2 \\
 & \qquad -5x^3 - 5x^2 \\
 & \qquad\quad 2x^2 + 2x \\
 & \qquad\quad 2x^2 + 2x \\
\hline
 & \qquad\qquad 0
\end{array}
$$

try root $(x-1)$

$$
\begin{array}{r|l}
 & 2x^3 + 3x^2 - 2x \\
\hline
x-1 & 2x^4 + x^3 - 5x^2 + 2x \\
 & 2x^4 - 2x^3 \\
 & \quad 3x^3 - 5x^2 \\
 & \quad 3x^3 - 3x^2 \\
 & \qquad -2x^2 + 2x \\
 & \qquad -2x^2 + 2x \\
\hline
 & \qquad\quad 0
\end{array}
$$

second root $= (x-1)$

$\downarrow$

left with $(x-1)(x+1)(2x^3+3x^2-2x)$

$$x(2x^2+3x-2)$$

$$(2x-1)(x+2)$$

Thus Ans: $x(x+1)(x-1)(2x-1)(x+2)$ #

<u>Partial fraction</u>

$$\frac{N(x)}{(ax+b)(cx+d)} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)}$$

$$\frac{N(x)}{(ax+b)^2} = \frac{A}{(ax+b)} + \frac{B}{(ax+b)^2}$$

$$\frac{N(x)}{(ax+b)(x^2+c)} \equiv \frac{A}{(ax+b)} + \frac{Bx+C}{(x^2+c)}$$

Ex,

$$\frac{5x+7}{x^3+2x^2-x-2}$$

$$= x^2(x+2)-(x+2)$$

$$= (x^2-1)(x+2)$$

$$\left[\frac{5x+7}{(x^2-1)(x+2)} = \frac{A}{x+2} + \frac{Bx+C}{(x^2-1)}\right](x^2-1)(x+2)$$

$$5x+7 = A(x^2-1)+(Bx+C)(x+2)$$

$$5x+7 = (A+B)x^2+(2B+C=5 \qquad 2C-A=7$$

$$A=-B,$$

Thus $-2(-2a + c = 5)$

$-A + 2c = 7$

$= 4a - \cancel{2c} = -10$

$-a + \cancel{2c} = 7$

$= 3a = -3$

$a = -1, \quad B = 1, \quad C = 3$

Answers $= \dfrac{-1}{(x+2)} + \dfrac{x+3}{(x^2-1)}$ #

Ex. 2

$$\frac{x^2+1}{x(x-1)^3} = \frac{A}{(x)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}$$

And solve

Ex. 3

$$\frac{x-3}{x(x+3)^2} = \frac{A}{x} + \frac{Bx+C}{(x+3)^2}$$

And solve for A, B, C

Ex 4. Find $\dfrac{x^4+x^3+x^2-x+1}{x(x^2+1)^2} = \dfrac{A}{x} + \dfrac{Bx+C}{(x^2+1)} + \dfrac{Dx+E}{(x^2+1)^2}$

and solve

Function

1. If f is even, then f' is odd

for even $f(-x) = f(x)$ ex: function $= x^2$

2. If f is odd, then $f' =$ even

for odd $f(-x) = -f(x)$ ex: function $= x^3$

Derivative

1. $f(x) = \sqrt{x}(1 - x)$

$= \sqrt{x} - x^{\frac{3}{2}}$

$f'(x) = \frac{1}{2\sqrt{x}} - \frac{3}{2}x^{\frac{1}{2}}$

$= \frac{1}{2\sqrt{x}} - \left(\frac{3\sqrt{x}}{2}\right)\sqrt{x}$

$= \frac{1}{2\sqrt{x}} - \frac{3x}{2\sqrt{x}} = \frac{1-3x}{2\sqrt{x}}$

2. $\frac{x^2 - 2\sqrt{x}}{x}$

$= \frac{(x)(2x - x^{-\frac{1}{2}}) - (x^2 - 2\sqrt{x})}{x^2}$

$= \frac{x(2x - \frac{1}{\sqrt{x}}) - (x^2 - 2\sqrt{x})}{x^2}$

$= \frac{2x^2 - \frac{x}{\sqrt{x}} - x^2 + 2\sqrt{x}}{x^2}$

$= \frac{x^2 - \sqrt{x} + 2\sqrt{x}}{x^2} = \frac{x^2 + \sqrt{x}}{x^2}$

$= \frac{\cancel{x^2}(1 + \frac{1}{\sqrt{x^3}})}{\cancel{x^2}}$

3. $f(x) = \sqrt{3}\,x - \sqrt{2x}$

$$= \sqrt{3} - \frac{1}{\cancel{2}}(2x)^{-\frac{1}{2}} \cdot \cancel{2}$$

$$= \sqrt{3} - \frac{1}{\sqrt{2x}}$$

4. $f(x) = \sqrt[3]{x^2} - \sqrt[2]{x^3}$

$$= x^{\frac{2}{3}} - x^{\frac{3}{2}}$$

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}} - \frac{3}{2}x^{\frac{1}{2}}$$

$$= \frac{2}{3\sqrt[3]{x}} - \frac{3\sqrt{x}}{2}$$

5. $f(x) = (x^2+1)\sqrt[3]{x^2+2}$

$$(x^2+2)^{\frac{1}{3}}(2x) + (x^2+1)\,\frac{1}{3}(x^2+2)^{-\frac{2}{3}} \cdot 2x$$

$$= 2x\sqrt[3]{x^2+2} + \frac{2x^3+2x}{3\sqrt[3]{(x^2+2)^2}}$$

$$= \frac{2x\,(x^2+2)^{\frac{1}{3}}\left(3\,(x^2+2)^{\frac{2}{3}}\right) + 2x^3+2x}{3\sqrt[3]{(x^2+2)^2}}$$

$$= \frac{6x\,(x^2+2) + 2x^4 + 2x}{3\sqrt[3]{(x^2+2)^2}} = \frac{6x^3 + 12x + 2x^3 + 2x}{3\sqrt[3]{(x^2+2)^2}}$$

$$= \frac{8x^3 + 14x}{3\sqrt[3]{(x^2+2)^2}}$$

$$f(x) = \sqrt{x + \sqrt{x + \sqrt{x}}}$$

$$= \left(x + (x+\sqrt{x})^{\frac{1}{2}}\right)^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}\left(x + (x+\sqrt{x})^{\frac{1}{2}}\right)^{-\frac{1}{2}} \cdot \left(1 + \left(\frac{1}{2}(x+\sqrt{x})^{-\frac{1}{2}} \cdot (1 + \frac{1}{2}x^{-\frac{1}{2}})\right)\right)$$

$$= \frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}} \cdot \left(1 + \left(\frac{1}{2\sqrt{x+\sqrt{x}}} \cdot \left(1 + \frac{1}{2\sqrt{x}}\right)\right)\right)$$

$$f(x) = \frac{(x-1)^4}{(x^2+2x)^5}$$

$$= \frac{(x^2+2x)^5 \cdot 4(x-1)^3 - (x-1)^4 \cdot 5(x^2+2x)^4 \cdot (2x+2)}{(x^2+2x)^{10}}$$

$$= \frac{4(x-1)^3}{(x^2+2x)^5} - \frac{(x-1)^4 \cdot (10x+10)}{(x^2+2x)^6}$$

$=$

nth derivative of a function

Note: $\left[(ax+b)^m\right]^{(n)} = \frac{m!}{(m-n)!}(ax+b)^{m-n} \cdot a^n$

Properties of exponential function

$$a^x = e^{x \ln a}$$

Ex. $\ln\left(a^{\frac{p}{q}}\right) = \cancel{\ln e}^{\frac{p}{q}\ln a}$

$$= \frac{p}{q}\ln a$$

* Remember

$\cos(0) = 1$, $\sin(0) = 0$

Additional

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

restrict $x + y \neq \frac{\pi}{2} + \pi k$

$x \neq \frac{\pi}{2} + \pi k$

$y \neq \frac{\pi}{2} + \pi k$

$$\sin(a+b) = \sin(a)\cos b + \cos(a)\sin(b)$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\sin^2(x) + \cos^2(x) = 1$$

Inverse Derivative

Ex: $f'(x)$ of $\arccos [-1, 1] \to [0, \pi]$

$= f(x) = \cos(x)$

$f'(x) = -\sin(x)$ $\quad f'(y) = -\sin(y)$ $\quad$ as $y = \cos(x)$, $x = \cos(y)$

$$f'(x)^{-1} = \frac{1}{f'(y)} = \frac{1}{-\sin(y)} = \frac{1}{-\sqrt{1-\cos^2 y}} = -\frac{1}{\sqrt{1-x^2}}$$

note $\sqrt{\sin^2 x} = \sqrt{1 - \cos^2 x}$

$\sin y = \sqrt{1 - \cos^2 y}$

Ex, 2 for $\arctan(x)$

$f(x) = \tan(x)$ $\quad$ $y = \tan x \Leftrightarrow x = \tan y$

$f'(x) = \sec^2(x)$ $\quad$ $f'(y) = \sec^2(y)$

$$f'(x)^{-1} = \frac{1}{f'(y)} = \frac{1}{\sec^2(y)} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

note $1 + \tan^2(y)$

as $\sec^2(x) = 1 + \tan^2(x)$

$Z \rightarrow z = r \cdot e^{i\theta}$

$= \cos(\theta) + i \sin\theta$

find $z^7 = 3+4i$

find r first $r = \sqrt{3^2+4^2}$

$r = 5$

$z = 5 \cdot \left(\cos(\underbrace{53}_{\frac{3}{5}}) + i \sin(\underbrace{53}_{\frac{4}{5}}) \right)$

find z^7: $k=0$; $z = \sqrt[7]{5} \left(\cos\left(\frac{53}{7} \cdot \frac{2\pi k}{7} \right) + i \sin\left(\frac{53}{7} \cdot \frac{2\pi k}{7} \right) \right)$

$k=1$ $z = \sqrt[7]{5} \left(\cos\left(\frac{53}{7} + \frac{2\pi(1)}{7} \right) + i \sin\left(\frac{53}{7} + \frac{2\pi(1)}{7} \right) \right)$

+ . . . . . . $\ddot{\text{ö}}$ $k = 7$

(ii) $z^2 - iz + 1 = 0$

$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= \frac{i \pm \sqrt{i^2-4}}{2}$

$= \frac{i \pm \sqrt{-5}}{2} = \frac{i \pm \sqrt{5}\, i}{2}$ //

(iii) $z^4 + z^2 + 1 = 0$

$= z^4 + 2z^2 + 1 - z = 0$

$(z^2+1)^2 - z^2 = 0$

$a^2 - b^2$

$((z^2+1)+z)((z^2+1)-z) = 0$

$\underbrace{z^2+1+z=0}_{\text{use factor and then done}}$ $\qquad$ $\underbrace{z^2+1-z=0}_{\text{use factor and done}}$

$\underbrace{\qquad\qquad}_{\text{4 answers}}$

$f(x) = \ln(1 + x^2)$

$f'(x) = \frac{1}{1+x^2} \cdot 2x$

$= \frac{2x}{1+x^2}$

$\lim_{x \to \infty} \frac{2x}{1+x^2}$

$\downarrow$

l'hopital Rule $= \frac{f'(x)}{g'(x)}$

$= \frac{2}{2x}$

$\lim_{x \to \infty} = \frac{2}{2(\infty)} = \frac{2}{\infty} = 0$ #

Inverse Derivative Problem

Practice Exam

Test for convergend $\displaystyle\sum_{K=1}^{\infty} \frac{5^K}{4^K+3^K}$

root test $= \sqrt[n]{|a_n|}$

$$= \sqrt[K]{\frac{5^K}{4^K+3^K}}$$

$$= \frac{5}{\sqrt[K]{4^K+3^K}} = \frac{5}{(4^K+3^K)^{\frac{1}{K}}} = \text{case } \frac{5}{\sqrt[K]{4^K}} = \frac{5}{4} > 1$$

as

so diverge

$$\sum_{k=1}^{\infty} \frac{3^k k^2}{k!} = \frac{\cancel{3^{(k+1)}}(k+1)^2}{(k+1)!} \cdot \frac{\cancel{k!}}{\cancel{3^k} k^2}$$

$$= \frac{3(k+1)^2}{\cancel{(k+1)}(k^2)} = \frac{3(k+1)}{k^2}$$

as k grow faster It will become larger than the numerator thus the ratio < 1.

$$= \frac{3\cdot\cancel{k^2}\left(\frac{1}{k}+\frac{1}{k^2}\right)}{\cancel{k^2}}$$

$$= 3\left(\frac{1}{k}+\frac{1}{k^2}\right)$$

$$\lim_{x\to\infty} = 3(0) = 0 < 1$$

so converge