Definition

Linear n^{th} order ODE in standard form:

$$L[y] := \frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = g(t), \quad t \in I, \quad (1)$$

where I is an open interval of $\mathbb{R}$. Initial condition:

$$y(t_0) = y_0,\ y'(t_0) = y_0',\ \ldots,\ y^{(n-1)}(t_0) = y_0^{(n-1)}, \quad (2)$$

where $t_0 \in I$ and $y_0, y_0', \ldots, y_0^{(n-1)} \in \mathbb{R}$.

Theorem: Existence and uniqueness

If functions $p_1, \ldots, p_n, g$ are continuous on the open interval I, then there exists exactly one solution $y = \phi(t)$ of the ODE (1) that also satisfies the initial conditions (2), where t_0 is any point in I. This solution exists throughout the interval I.

Theorem

Theorem

Let $p_1, \ldots, p_n$ be continuous on the open interval I, and $y_1, \ldots, y_n$ be solutions of the homogeneous equation (3):

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0.$$

If there exists $t_0 \in I$ such that $W[y_1, \ldots, y_n](t_0) \neq 0$, then every solution of (3) can be expressed as a linear combination of the solutions $y_1, \ldots, y_n$:

$$y(t) = c_1 y_1(t) + \cdots + c_n y_n(t).$$

This is the **general solution**, and $y_1, \ldots, y_n$ form a **fundamental set of solutions**.

CH 5: Linear System of ODEs

Convert n-th order equation to a system of 1st Order ODE

Convert $Mu'' + ru' + Ku = F(t)$

Let $y_1 = u,\ y_1' = y_2,\ u' = y_2 \Rightarrow u'' = y_2'$

Hence $\rightarrow$ $My_2' + ry_2 + Ky_1 = F(t)$

Now as $y_1' = y_2$

$y_2' = \frac{F(t)}{M} - \frac{r}{M}y_2 - \frac{K}{M}y_1$

Convert to matrix: $y_1\begin{vmatrix} 0 \\ -\frac{K}{M}\end{vmatrix} + y_2\begin{vmatrix} 1 \\ -\frac{r}{M}\end{vmatrix} + 1\begin{vmatrix} 0 \\ \frac{F(t)}{M}\end{vmatrix}$

Hence $\Rightarrow$ $x' = Az + y$, $A = \begin{vmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{r}{M}\end{vmatrix}$, $y = \begin{pmatrix} 0 \\ \frac{F(t)}{M}\end{pmatrix}$

Theorem: Existence and Uniqueness - Nonlinear case

Let each of the n functions $F_1, \ldots, F_n$ and their n^2 first partial derivatives

$$\partial_{x_1}F_1, \ldots, \partial_{x_n}F_1, \ldots, \partial_{x_1}F_n, \ldots, \partial_{x_n}F_n$$

be continuous in a region R of $tx_1x_2\ldots x_n$-space defined by:

$$\alpha < t < \beta,\ \alpha_1 < x_1 < \beta_1,\ \ldots,\ \alpha_n < x_n < \beta_n,$$

and let the point determining the initial condition belong to that region:

$$(t_0, x_1^0, x_2^0, \ldots, x_n^0) \in R = (\alpha, \beta) \times (\alpha_1, \beta_1) \times \cdots \times (\alpha_n, \beta_n).$$

THEN there exists an interval $(t_0 - h, t_0 + h)$ in which there exists a unique solution of the initial value problem.

Remark

If their function and first partial derivative are smooth and continuous in certain region. Then, there gurantee a unique solution to the diff eq atleast for a small time interval.

Theorem: Existence and Uniqueness - Linear case

Let $I = (\alpha, \beta)$ be an open interval of $\mathbb{R}$ over which the functions

$$p_{11}, p_{12}, \ldots, p_{nn}, g_1, \ldots, g_n$$

are all continuous. Let $t_0 \in I$, and $x_1^0, \ldots, x_n^0 \in \mathbb{R}$. Then the initial value problem equation (18) coupled with

$$(x_1(t_0), \ldots, x_n(t_0)) = (x_1^0, \ldots, x_n^0)$$

admits a unique solution, which is defined throughout the whole interval I.

Linear Dependent / Linear Independent.

Theorem 7.1 $\Rightarrow$ If the vector functions $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are solutions of the system (3), then the linear combination $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$ is also a solution for any constants c_1 and c_2.

Can be proved by differentiating $c_1x^{(1)} + c_2x^{(2)}$

Given $X(t) \Rightarrow \begin{pmatrix} x_{11}(t) & \cdots & x_{1n}(t) \\ \vdots & & \vdots \\ x_{n1}(t) & \cdots & x_{nn}(t)\end{pmatrix}$ Linearly independent for a given value of t if $\operatorname{Det} X(t) \neq 0$ Wronskian.

Theorem 7.4.2 If the vector functions $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ are linearly independent solutions of the system (3) for each point in the interval $\alpha < t < \beta$, then each solution $\mathbf{x} = \boldsymbol{\phi}(t)$ of the system (3) can be expressed as a linear combination of $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$

$$\boldsymbol{\phi}(t) = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t) \quad (11)$$

in exactly one way.

All sol of $x' = P(t)x$ can be write as: General Sol.

Fundamental Set of Sol $\Rightarrow$ Set of Sol; $x^{(1)}, x^{(2)}, \ldots x^{(n)}$ that is linearly independent at every point $t \in (\alpha, \beta)$

Theorem: (Abel)

If $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}$ are solutions of the homogeneous linear system of n first-order ODEs

$$x' = P(t)x, \quad t \in I = (\alpha, \beta),$$

then their Wronskian $W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}]$ either is identically zero over I, or never vanishes over I.

$\rightarrow$ Don't need to find $W(x)$ at all p.n.t in the interval of interest, enable us to determine whether $x^{(1)} \ldots x^{(n)}$ form a fundamental set of Sol.

Eigen Value / Eigen Vector.

The Eigen Value of A are the real number λ such that $= \lambda^2 - (\operatorname{Tr}(A))\lambda + \operatorname{Det}(A)$

$Av = \lambda v$ $\rightarrow$ Find Eigen Vector use $(A - \lambda I)\vec{v} = 0$

Admits atleast one non-trivial solution $v \neq \vec{0}$

Every such $v \neq 0$ is called an Eigen vector of A

The eigenspace associated to λ, Eigen value is the span of all eigen vector associated to λ.

Can also find λ by using

Prove using existence and uniqueness.

► Show that for any $x_0 \in \mathbb{R}^2$ there exist $c_1, c_2 \in \mathbb{R}$ such that $\vec{x_0} = c_1\vec{v_1} + c_2\vec{v_2}$: for any initial condition, there's a solution of the form $c_1\vec{v_1} + c_2\vec{v_2}$ that takes it.

let $\begin{pmatrix} V_{11} & V_{21} \\ V_{12} & V_{22}\end{pmatrix}\begin{pmatrix} C_1 \\ C_2\end{pmatrix} = \begin{pmatrix} X_{01} \\ X_{02}\end{pmatrix}$; also $\det(\vec{v_1}, \vec{v_2}) \neq 0$

If $\det \neq 0$ then there exist a unique sol in the open interval I.

Hence $\exists$; sol $\Rightarrow \begin{pmatrix} C_1 \\ C_2\end{pmatrix}$ #

► Use the existence and uniqueness result to conclude that all solutions can be written in the form $c_1\vec{v_1} + c_2\vec{v_2}$.

Define $x^{(1)}t = \vec{v}_1e^{\lambda_1 t}$ and $x^{(2)}t = \vec{v}_2e^{\lambda_2 t}$

Check if $x^{(1)}t, x^{(2)}t$ satisfy the diff eq. $\rightarrow$ Prove by differentiating

$Ax^{(1)}(t) = A\vec{v}_1e^{\lambda_1 t} = \lambda_1\vec{v}_1e^{\lambda_1 t} \Rightarrow x^{(1)'}t$

$Ax^{(2)}t = A\vec{v}_2e^{\lambda_2 t} = \lambda_2\vec{v}_2e^{\lambda_2 t} \Rightarrow x^{(2)'}t$

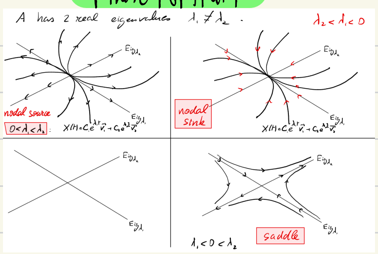
$x^{(1)'} = x^{(1)}$ ✓ Prove sol

Let General eq. $\Rightarrow$ $x(t) \Rightarrow c_1x^{(1)}t + c_2x^{(2)}t$

$= c_1\vec{v}_1e^{\lambda_1 t} + c_2\vec{v}_2e^{\lambda_2 t}$

Initial Value problem $\Rightarrow$ $x(0) = c_1\vec{v}_1 + c_2\vec{v}_2$

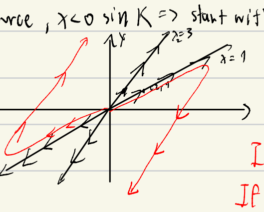
Phase Portrait



Phase portrait ex: for $A = \begin{bmatrix} 1 & 1 \\ -2 & 5\end{bmatrix}$, $\lambda_1 = 1$, $\lambda_2 = 3$

Step 1 Start with plot vector $v_1 = \begin{pmatrix} 2 \\ 1\end{pmatrix}$ $v_2 = \begin{pmatrix} 1 \\ 1\end{pmatrix}$

2: $\lambda > 0$ source, $\lambda < 0$ sink $\Rightarrow$ start with smallest λ to highest



Complex Value. $\rightarrow$ Choose point to find direction

for eigen value $\Rightarrow N \pm 6i$

If $N > 0$ spiral source (out)

If $N < 0$ spiral sink (Inward)

If $N = 0$ then it is centre (circle)/ellipse.

Diagonalization.

$A = PDP^{-1}$ $\rightarrow$ D = Diagonal Matrix, P = Matrix of eigen vector A

$P^{-1} \Rightarrow$ If $P = \begin{vmatrix} a & b \\ c & d\end{vmatrix} \rightarrow P^{-1} = \begin{vmatrix} d & -b \\ -c & a\end{vmatrix}\frac{1}{\det(P)}$

Original Matrix $\Rightarrow$ Matrix A is diagonalizable if it has n linearly independent vector and has complete set of eigen value.

Real Repeated Eigen Value.

Suppose got $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1\end{pmatrix}$ Use row reduct $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{pmatrix}(\vec{v}_n)$

Hence $\vec{v}_1 + \vec{v}_2 + \vec{v}_3 = 0$ let $v_2 = \alpha$, $v_3 = \beta$ then $v = \begin{pmatrix} -\alpha - \beta \\ \alpha \\ \beta\end{pmatrix}$

vector then $\Rightarrow$ $v_1\begin{pmatrix} -1 \\ 1 \\ 0\end{pmatrix} + \vec{v}_2\begin{pmatrix} -1 \\ 0 \\ 1\end{pmatrix}$

Hard Case general sol $\Rightarrow$ $x(t) = c_1e^{\lambda t}v + c_2(te^{\lambda t}v + e^{\lambda t}W)$

How to find $W \rightarrow (A - \lambda I)W = \vec{v}$

Solution for Complex Eigen Value

$x(t) = e^{\lambda t}v \Rightarrow e^{(N \pm 6i)t}(u \pm iw)$

$x^{(1)}t = e^{Nt}(\cos(6t)u - \sin(6t)w)$

$x^{(2)}t = e^{Nt}(\sin(6t)u + \cos(6t)w)$

General sol $\rightarrow$ $x(t) = e^{Nt}(c_1(\cos(6t))v_1 + c_2(\sin 6t)v_2)$

$x^{(1)}(t) = e^{\mu t}(\cos(\sigma t)u - \sin(\sigma t)w)$, $x^{(2)}(t) = e^{\mu t}(\sin(\sigma t)u + \cos(\sigma t)w)$. $\rightarrow \begin{cases} x^{(1)}(0) = \vec{u} \\ x^{(2)}(0) = \vec{w}\end{cases}$ $\rightarrow$ Prove that for any initial condition we can find sol in $c_1x^{(1)}t + c_2x^{(2)}t$

$\vec{u}, \vec{w}$ linearly independent implies that $\operatorname{Span}_{\mathbb{R}}(\vec{u}, \vec{w}) = \mathbb{R}^2 = \operatorname{Span}_{\mathbb{R}}(\{(\begin{smallmatrix}1\\0\end{smallmatrix}), (\begin{smallmatrix}0\\1\end{smallmatrix})\})$

Hence, for any $\vec{x}_0 = \begin{pmatrix} x_{01} \\ x_{02}\end{pmatrix} \in \mathbb{R}^2$, we have a linear combination:

$c_1x^{(1)}(0) + c_2x^{(2)}(0) = c_1\vec{u} + c_2\vec{w} = \begin{pmatrix} x_{01} \\ x_{02}\end{pmatrix} = \vec{x}_0$

System of First Order Linear ODE $\Rightarrow$ $X' = A(t)X$

$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t)\end{pmatrix}$ $a_{11}(t) - a_{22}(t)$ are coefficient Homogeneous.

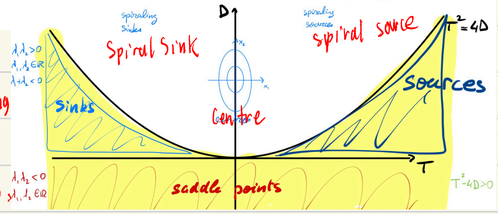
Trace Determinant Plane.

Recall that, for $A \in M_2(\mathbb{R})$, $\operatorname{Tr}(A)$, $D = \operatorname{Det}(A)$

$$\det(A - \lambda I_2) = \lambda^2 - \operatorname{Tr}(A)\lambda + \det(A) = \lambda^2 - (a+d)\lambda + \lambda d$$

$A\begin{pmatrix} a & b \\ c & d\end{pmatrix}$: $\operatorname{Tr}(A) = a + d$ (trace), $\operatorname{Det}(A) = ad - bc$

$\lambda = \frac{1}{2}T \pm \frac{1}{2}\sqrt{T^2 - 4D}$



Stability and linearisation.

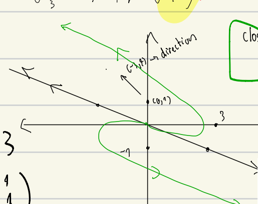
equilibrium Sol $\Rightarrow$ A point Consider equilibrium sol when if $f(x_0, y_0) = g(x_0, y_0) = 0$ or when $x'(t) = 0$, $y'(t) = 0$

Real Repeated Value $A = \begin{pmatrix} 2 & -3 \\ \frac{1}{3} & 4\end{pmatrix}$

$\lambda = 3$ $\vec{v} = \begin{pmatrix} 3 \\ -1\end{pmatrix}$

Choose point $(0\ 1)$

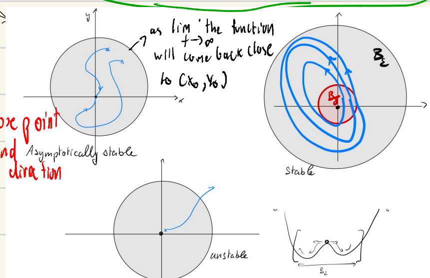
$\begin{pmatrix} 2 & -3 \\ \frac{1}{3} & 4\end{pmatrix}\begin{pmatrix} 0 \\ 1\end{pmatrix} = \begin{pmatrix} -3 \\ 4\end{pmatrix}$ Direction



Stable $\Rightarrow$ if $\vec{x}(t)$ stays close to $\vec{x}_0$ for all $t > 0$

Asymptotically Stable $\Rightarrow$ If $\vec{x}(t)$ stay close to $\vec{x}_0$ for all $t > 0$ and also $\lim_{t \to \infty} \vec{x}(t) = \vec{x}_0$.

Unstable $\Rightarrow$ If $\vec{x}(t)$ leaves the neighborhood of $\vec{x}_0$.



How to find Critical Pnt

Ex. given $x' = x(2 - x - y)$, $y' = (1 - y)(2 + x)$

ex: case 1 $x = 0$, $1 - y = 0$ $\rightarrow$ $x = 0, y = 1$

ex: case 2 $x = 0$, $2 + x = 0$ $\rightarrow$ no sol

STABILITY

2 Distinct Real Eigen Value.

If $\lambda_1 < \lambda_2 < 0$ $\rightarrow$ $e^{\lambda_1 t}, e^{\lambda_2 t} \rightarrow 0$ as $t \rightarrow \infty$ Hence it is asymptotically stable

If $\lambda_1 < 0, \lambda_2 > 0$ $\rightarrow$ unstable

one repeated real eigen value

$x(t) = c_1e^{\lambda t}v + c_2te^{\lambda t}W$

If $\lambda < 0$ $\rightarrow$ asymptotically stable

If $\lambda = 0$ $\rightarrow$ unstable

If $\lambda > 0$ $\rightarrow$ unstable

Complex Conjugate eigen Value

If $N < 0$ $\rightarrow$ stable

Close loop

If $N = 0$ $\rightarrow$ stable, If $N > 0$ - unstable

Definition: Periodic solutions and cycles

A solution $(x, y)^T(t)$ is called **periodic** with period $T > 0$ if for all t, (Solution repeat after interval)

$$x(t+T) = x(t), \quad \text{and} \quad y(t+T) = y(t).$$

Hence, a periodic solution is defined for all t. Note that an equilibrium solution

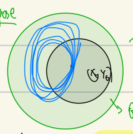
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \equiv \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

is periodic with period T for any $T > 0$. A periodic solution that is not an equilibrium solution is called a **cycle**. → Cycle → close loop by a factor of $n\pi$

$x_1 = y, \; x_2 = y'$ ←

Exercise. => Show that cycles are stable, but not asymptotically stable

Suppose



It is stable as there exist a smaller neighborhood B_δ such that the trajectory originate from B_δ is inside the neighbor hood of $B_\varepsilon(x_0 y_0)$

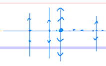
– It is not asymptically stable because the solution will repeated itself by a factor of $n\pi$ such that as $t \to \infty$ it will never stop exactly or close to the neighbour hood where it originated.

Note => Cycle exist if $a + d = 0$ and $ad - bc > 0$

Definition: Trapped trajectories

Let R be a bounded, connected region of $\mathbb{R}^2$. We say that R **traps trajectories** of the system (2) if any trajectory $\mathbf{x}$ originating at $\mathbf{x}(t_0) \in R$ remains in R for all times $t \geq t_0$.

$\vec{x}' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\vec{x}$ | $\vec{x}' = \begin{pmatrix} 0 & 0 \\ 0 & \pm 1 \end{pmatrix}\vec{x}$



Linearisation

Theorem: Linear approximation

Assume that (x_0, y_0) is an **isolated equilibrium solution**, *i.e.* there are no other equilibrium solutions in the vicinity of (x_0, y_0). Define

$$A = \begin{pmatrix} \partial_x f(x_0, y_0) & \partial_y f(x_0, y_0) \\ \partial_x g(x_0, y_0) & \partial_y g(x_0, y_0) \end{pmatrix}. \qquad \vec{x}' = A\vec{x}$$

The stability properties of (3) are related to those of the linear approximation $\vec{x}' = A\vec{x}$ as follows:

- If A has real eigenvalues $\lambda_1 < 0, \lambda_2 < 0$, then the solution (x_0, y_0) is asymptotically stable.
- If $\lambda_1 > 0$ or $\lambda_2 > 0$, then (x_0, y_0) is unstable.
- If A has complex eigenvalues $\mu \pm i\sigma$, if $\mu < 0$, then (x_0, y_0) is asymptotically stable, and if $\mu > 0$, then (x_0, y_0) is unstable.

!! If the eigen value of linearlied matrix are purely imaginary then using linearisation might not be that helpful

Lyapunov Function => $(x(t))^2 + (y(t))^2$

Predator / Prey system, Competing species

Competing Species. => Without Competition population of each species (x,y)

$$\frac{dx}{dt} = x(d_1 - 6_1 x), \quad \frac{dy}{dt} = y(d_2 - 6_2 y)$$ } (Logistic Growth (Individual Species))

d_1, d_2 = intrinsic growth rate, $6_1, 6_2$ => saturation levels of each population.

With Competition =>

$$\frac{dx}{dt} = x(d_1 - 6_1 x - d_1 y), \quad \frac{dy}{dt} = y(d_2 - 6_2 y - d_2 x)$$

d_1, d_2 => Coefficient measuring the effect of one species on the growth of the other.

How to find Critical P.n.t =>

$$x(d_1 - 6_1 x - d_1 y) = 0, \quad y(d_2 - 6_2 y - d_2 x) = 0$$

Predator Prey Model. Prey => $\frac{dx}{dt} = ax - \alpha xy \rightarrow x(a - \alpha y)$

x = prey population, a = Prey growth rate, α (alpha) => Predation rate coefficient.

Predator => $\frac{dy}{dt} = -cy + \gamma xy$ → y = Predator population, c = predator death rate,

γ: Predator growth rate due to successful predation.

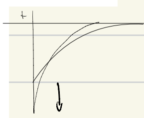
Assumption.

In the absence of the predator, the prey grows at a rate proportional to the current population; thus $dx/dt = ax, a > 0$, when $y = 0$.

In the absence of the prey, the predator dies out; thus $dy/dt = -cy, c > 0$, when $x = 0$.

The number of encounters between predator and prey is proportional to the product of their populations. Each such encounter tends to promote the growth of the predator and to inhibit the growth of the prey. Thus the growth rate of the predator is increased by a term of the form γxy, while the growth rate of the prey is decreased by a term $-\alpha xy$, where γ and α are positive constants.

28. Consider a 2×2 system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. If we assume that $r_1 \neq r_2$, the general solution is $\mathbf{x} = c_1\boldsymbol{\xi}^{(1)}e^{r_1 t} + c_2\boldsymbol{\xi}^{(2)}e^{r_2 t}$, provided that $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ are linearly independent. In this problem we establish the linear independence of $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ by assuming that they are linearly dependent and then showing that this leads to a contradiction.
(a) Note that $\boldsymbol{\xi}^{(1)}$ satisfies the matrix equation $(\mathbf{A} - r_1\mathbf{I})\boldsymbol{\xi}^{(1)} = \mathbf{0}$; similarly, note that $(\mathbf{A} - r_2\mathbf{I})\boldsymbol{\xi}^{(2)} = \mathbf{0}$.
(b) Show that $(\mathbf{A} - r_2\mathbf{I})\boldsymbol{\xi}^{(1)} = (r_1 - r_2)\boldsymbol{\xi}^{(1)}$.
(c) Suppose that $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ are linearly dependent. Then $c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)} = \mathbf{0}$ and at least one of c_1 and c_2 (say c_1) is not zero. Show that $(\mathbf{A} - r_2\mathbf{I})(c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)}) = \mathbf{0}$, and also show that $(\mathbf{A} - r_2\mathbf{I})(c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)}) = c_1(r_1 - r_2)\boldsymbol{\xi}^{(1)}$. Hence $c_1 = 0$, which is a contradiction. Therefore, $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ are linearly independent.
(d) Modify the argument of part (c) if we assume that $c_2 \neq 0$.
(e) Carry out a similar argument for the case in which the order n is equal to 3; note that the procedure can be extended to an arbitrary value of n.



Graph for tank Q.s.

(b.) $(A - r_2 I)\xi^{(1)} = A\xi^{(1)} - r_2 I \xi^{(1)}$
as $A\xi^{(1)} = r_1 \xi^{(1)}$
then $r_1\xi^{(1)} - r_2\xi^{(1)} \Rightarrow (r_1 - r_2)\xi^{(1)}$ # Prove

(c.) ↘ for first one => Assume that $c_1 \neq 0$ for linearly dependent then it is clear that $(A - r_2 I)(c_1\xi^1 + c_2\xi^2) = 0$ → linear dependent, there exist a constant such that at least either c_1 or $c_2 \neq 0$

$(A - r_2 I)(c_1\xi^{(1)} + c_2\xi^{(2)})$
$Ac_1\xi^{(1)} + Ac_2\xi^{(2)} - r_2 c_1\xi^{(1)} - r_2 c_2\xi^{(2)}$
$c_1 r_1\xi^{(1)} + r_2 c_2\xi^{(2)} - r_2 c_1\xi^{(1)} - r_2 c_2\xi^{(2)} = 0$
$c_1(r_1 - r_2)\xi^{(1)} = 0$
$c_1 = 0$

$$y'' + p(t)y' + q(t)y = 0 \qquad \text{(i)}$$

corresponds to the system

$$x_1' = x_2,$$
$$x_2' = -q(t)x_1 - p(t)x_2. \qquad \text{(ii)}$$

Show that if $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are a fundamental set of solutions of Eqs. (ii), and if $y^{(1)}$ and $y^{(2)}$ are a fundamental set of solutions of Eq. (i), then $W[y^{(1)}, y^{(2)}] = cW[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}]$, where c is a nonzero constant.
Hint: $y^{(1)}(t)$ and $y^{(2)}(t)$ must be linear combinations of $x_{11}(t)$ and $x_{12}(t)$.

5. Show that the general solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$ is the sum of any particular solution $\mathbf{x}^{(p)}$

Since both of the pair are solution to the same differential equation. Hence $y^{(1)}$ and $y^{(2)}$ must be a linear combination of the solution x_1.
let $y^{(1)} = q_{11}x_{11} + q_{12}x_{12}$
$y^{(2)} = q_{21}x_{11} + q_{22}x_{12}$
Where a is also a constant => $W(y^{(1)}, y^{(2)}) \Rightarrow \begin{vmatrix} y^{(1)} & y^{(2)} \\ y'^{(1)} & y'^{(2)} \end{vmatrix}$

$$W = \begin{vmatrix} q_{11}x_{11} + q_{12}x_{12} & q_{21}x_{11} + q_{22}x_{12} \\ q_{11}x'_{11} + q_{12}x'_{12} & q_{21}x'_{11} + q_{22}x'_{12} \end{vmatrix}$$

$W = (q_{11}q_{22} - q_{12}q_{21})x_{11}x'_{12} - (q_{11}q_{22} - q_{12}q_{21})x_{12}x'_{11}$

$x_2 = y' = x'_1$

Hence $W = (q_{11}q_{22} - q_{12}q_{21})x_{11}x_{22} - (q_{11}q_{22} - q_{12}q_{21})x_{12}x_{21}$

$(q_{11}q_{22} - q_{12}q_{21})(x_{11}x_{22} - x_{12}x_{21})$

$\rightarrow (q_{11}q_{22} - q_{12}q_{21}) \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}$

Prove → $W(y^{(1)}, y^{(2)}) = cW(x^{(1)}, x^{(2)})$

8. Let $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}$ be solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on the interval $\alpha < t < \beta$. Assume that $\mathbf{P}$ is continuous, and let t_0 be an arbitrary point in the given interval. Show that $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}$ are linearly dependent for $\alpha < t < \beta$ if (and only if) $\mathbf{x}^{(1)}(t_0), \ldots, \mathbf{x}^{(m)}(t_0)$ are linearly dependent. In other words $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}$ are linearly dependent on the interval if they are linearly dependent at any point in it.
Hint: There are constants $c_1, \ldots, c_m$ that satisfy $c_1\mathbf{x}^{(1)}(t_0) + \cdots + c_m\mathbf{x}^{(m)}(t_0) = \mathbf{0}$. Let $\mathbf{z}(t) = c_1\mathbf{x}^{(1)}(t) + \cdots + c_m\mathbf{x}^{(m)}(t)$, and use the uniqueness theorem to show that $\mathbf{z}(t) = \mathbf{0}$ for each t in $\alpha < t < \beta$.

As $x^{(1)}(t_0), \ldots x^{(m)}(t_0)$ are linearly independent at t_0. This means there exist a constant $c_1, \ldots, c_m$ not all zero such that
$c_1 x^{(1)}(t_0) + c_2 x^{(2)}(t_0) + \cdots c_m x^{(m)}(t_0) = 0$

Define => $z(t) = c_1 x^{(1)}(t) + c_2 x^{2}t + c_m x^{(m)}(t_0)$ → Hence also $z(t_0)$ should be $= 0$

$z'(t) = c_1 x'^{(1)}(t) + c_2 x'^{(2)}(t) + \cdots c_m x'^{(m)}(t)$

$x'^{(1)}(t) = P(t)x^{(1)}$; As $x' = P(t)x(t)$

$z'(t) = c_1 (P(t)x^{(1)}(t)) + c_2 P(t)x^{(2)}(t)$

$= z'(t) = P(t)(c_1 x^{(1)}(t) + c_2 x^{(2)}(t))$

Satisfy as $z'(t) = P(t)z(t)$ <=> $x' = P(t)x$

(a) Show that any solution $\mathbf{x} = \mathbf{z}(t)$ can be written in the form

$$\mathbf{z}(t) = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t)$$

for suitable constants $c_1, \ldots, c_n$.
Hint: Use the result of Problem 12 of Section 7.3, and also Problem 8 above.
(b) Show that the expression for the solution $\mathbf{z}(t)$ in part (a) is unique; that is, if $\mathbf{z}(t) = k_1\mathbf{x}^{(1)}(t) + \cdots + k_n\mathbf{x}^{(n)}(t)$, then $k_1 = c_1, \ldots, k_n = c_n$.
Hint: Show that $(k_1 - c_1)\mathbf{x}^{(1)}(t) + \cdots + (k_n - c_n)\mathbf{x}^{(n)}(t) = \mathbf{0}$ for each t in $\alpha < t < \beta$, and use the linear independence of $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$.

(a.) As $z(t) = c_1 x^{(1)}t + \cdots + c_n x^{(n)}t$
then ↘ $z'(t) = c_1 x'^{(1)}t + \cdots c_n x'^{(n)}t$
We know that ↘ $x'(t) = P(t)x$ so → $z'(t) = c_1 P(t)x^1 + c_n P(t)x^{(n)}$
$z'(t) = P(t)(c_1 x^1 + c_n x^n)$
$= z'(t) = P(t)z(t)$
Prove.

(b.) → We know that → $z(t) = c_1 x^{(1)}(t) + \cdots c_n x^{(n)}(t)$
also $z(t) = k_1 x^{(1)}t + \cdots k_n x^{(n)}t$ Apply initial condition $z(t_0) =$
As both of eq are solution hence => $c_1 x^{(1)}t + \cdots c_n x^{(n)}t = k_1 x^{(1)}t + \cdots k_n x^{(n)}t$
then $(c_1 - k_1)x^{(1)}t + \cdots (c_n - k_n)x^{(n)}t = 0$
Since it is linearly independent then $c_i - k_i = 0$ → for all $i = 1, 2, \ldots$

Circuit real Value question

$= \lambda^2 + \left(\frac{R_1}{L} + \frac{1}{CR_2}\right)\lambda + \left(\frac{R_1}{LCR_2} + \frac{1}{CL}\right)$

for Eigen Value to be real then $b^2 - 4ac > 0$

$\left(\frac{R_1}{L} + \frac{1}{CR_2}\right)^2 - 4(1)\left(\frac{R_1}{LCR_2} + \frac{1}{CL}\right) > 0$

$\frac{R_1^2}{L^2} + \frac{2R_1}{LCR_2} + \frac{1}{(CR_2)^2} - \frac{4R_1}{LCR_2} - \frac{4}{CL} > 0$

$\frac{R_1^2}{L^2} - \frac{2R_1}{LCR_2} + \frac{1}{C^2R_2^2} - \frac{4}{CL} > 0$

Note if we use Case $\lambda = -1 - \sqrt{2}i$ then the form is

$$x^{(2)}t = e^{-t}(\cos(\sqrt{2}t) - i\sin(\sqrt{2}t))\begin{pmatrix} -\sqrt{2}i \\ 1 \\ \sqrt{2}i - 1 \end{pmatrix}$$

for whole Solution

Check both eigen Value negative.

from Characteristic eq => $\lambda^2 - Tr(A)\lambda + Det(A) = 0$
We know $\lambda_1 + \lambda_2 = Tr(A)$ and $\lambda_1 \cdot \lambda_2 = Det(A)$

$Tr(A) = \left(-\frac{R_1}{L} - \frac{1}{CR_2}\right)$ } negative ; $Det(A) = \left(\frac{R_1}{L} \times \frac{1}{CR_2} + \frac{1}{CL}\right)$ } Positive

Hence $-\lambda_1 + (-\lambda_2)$ = negative $\quad -\lambda_1 \cdot -\lambda_2$ = Positive

As $\lambda_1 < \lambda_2 < 0$ → It is nodal sink; Hence Stable at $(0,0)$.

Voltage Q.S.

20. The characteristic equation is $r^2 + 2r - (24 + 8\alpha) = 0$, with roots

$$r_{1,2} = -1 \pm \sqrt{25 + 8\alpha}.$$

The roots are complex when $\alpha < -25/8$. Since the real part is negative, the origin is a stable spiral. Otherwise the roots are real. When $-25/8 < \alpha < -3$, both roots are negative, and hence the equilibrium point is a stable node. For $\alpha > -3$, the roots are of opposite sign and the origin is a saddle.

(b) With the specified values, the eigenvalues are $r_{1,2} = -1 \pm i$. The eigenvector corresponding to $r = -1 + i$ is $\boldsymbol{\xi}^{(1)} = (1, -4i)^T$. Hence one complex-valued solution is

$$\begin{pmatrix} I \\ V \end{pmatrix}^{(1)} = \begin{pmatrix} 1 \\ -1+i \end{pmatrix}e^{(-1+i)t} = \begin{pmatrix} 1 \\ -1+i \end{pmatrix}e^{-t}(\cos t + i\sin t) =$$
$$= e^{-t}\begin{pmatrix} \cos t \\ -\cos t - \sin t \end{pmatrix} + ie^{-t}\begin{pmatrix} \sin t \\ \cos t - \sin t \end{pmatrix}.$$

27.(a) Suppose that $c_1\mathbf{a} + c_2\mathbf{b} = \mathbf{0}$. Since $\mathbf{a}$ and $\mathbf{b}$ are the real and imaginary parts of the vector $\boldsymbol{\xi}^{(1)}$, respectively, $\mathbf{a} = (\boldsymbol{\xi}^{(1)} + \overline{\boldsymbol{\xi}^{(1)}})/2$ and $\mathbf{b} = (\boldsymbol{\xi}^{(1)} - \overline{\boldsymbol{\xi}^{(1)}})/2i$. Hence

$$c_1(\boldsymbol{\xi}^{(1)} + \overline{\boldsymbol{\xi}^{(1)}}) - ic_2(\boldsymbol{\xi}^{(1)} - \overline{\boldsymbol{\xi}^{(1)}}) = \mathbf{0},$$

which leads to

$$(c_1 - ic_2)\boldsymbol{\xi}^{(1)} + (c_1 + ic_2)\overline{\boldsymbol{\xi}^{(1)}} = \mathbf{0}.$$

Complex ↑ linear independent

(c) Recall that

$$\mathbf{u}(t) = e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t)$$
$$\mathbf{v}(t) = e^{\lambda t}(\mathbf{a}\cos\mu t + \mathbf{b}\sin\mu t).$$

Consider the equation $c_1\mathbf{u}(t_0) + c_2\mathbf{v}(t_0) = \mathbf{0}$, for some t_0. We can then write

$$c_1 e^{\lambda t_0}(\mathbf{a}\cos\mu t_0 - \mathbf{b}\sin\mu t_0) + c_2 e^{\lambda t_0}(\mathbf{a}\cos\mu t_0 + \mathbf{b}\sin\mu t_0) = \mathbf{0}. \; (*)$$

Rearranging the terms, and dividing by the exponential,

$$(c_1 + c_2)\cos\mu t_0\,\mathbf{a} + (c_2 - c_1)\sin\mu t_0\,\mathbf{b} = \mathbf{0}.$$

From part (b), since $\mathbf{a}$ and $\mathbf{b}$ are linearly independent, it follows that

$$(c_1 + c_2)\cos\mu t_0 = (c_2 - c_1)\sin\mu t_0 = 0.$$

Without loss of generality, assume that the trigonometric factors are nonzero. Otherwise proceed again from Equation (*), above. We then conclude that

$$c_1 + c_2 = 0 \text{ and } c_2 - c_1 = 0,$$

which leads to $c_1 = c_2 = 0$. Thus $\mathbf{u}(t_0)$ and $\mathbf{v}(t_0)$ are linearly independent for some t_0, and hence the functions are linearly independent at every point.

(d) The general solution of the second order equation is

$$u(t) = c_1\cos\sqrt{\frac{k}{m}}t + c_2\sin\sqrt{\frac{k}{m}}t.$$

The general solution of the system of ODEs is given by

$$\mathbf{x} = c_1\begin{pmatrix} \sqrt{\frac{m}{k}}\sin\sqrt{\frac{k}{m}}t \\ \cos\sqrt{\frac{k}{m}}t \end{pmatrix} + c_2\begin{pmatrix} \sqrt{\frac{m}{k}}\cos\sqrt{\frac{k}{m}}t \\ -\sin\sqrt{\frac{k}{m}}t \end{pmatrix}.$$

It is evident that the natural frequency of the system is equal to $|r_1| = |r_2|$.

Spring mass

8. Two species of fish that compete with each other for food, but do not prey on each other, are bluegill and redear. Suppose that a pond is stocked with bluegill and redear, and let x and y be the populations of bluegill and redear, respectively, at time t. Suppose further that the competition is modeled by the equations

$$dx/dt = x(\epsilon_1 - \sigma_1 x - \alpha_1 y),$$
$$dy/dt = y(\epsilon_2 - \sigma_2 y - \alpha_2 x).$$

(a) If $\epsilon_2/\alpha_2 > \epsilon_1/\sigma_1$ and $\epsilon_2/\sigma_2 > \epsilon_1/\alpha_1$, show that the only equilibrium populations in the pond are no fish, no redear, or no bluegill. What will happen for large t?
(b) If $\epsilon_1/\sigma_1 > \epsilon_2/\alpha_2$ and $\epsilon_1/\alpha_1 > \epsilon_2/\sigma_2$, show that the only equilibrium populations in the pond are no fish, no redear, or no bluegill. What will happen for large t?

$$dx/dt = x(1 - x - y),$$
$$dy/dt = y(0.5 - 0.25y - 0.75x),$$

↓ Competing Species ex.

Linearisation.

Suppose $x'(t) = x(2 - x - y)$ → $f(x)$
$y'(t) = -x + 3y - 2xy$ → $g(x)$

$$\begin{vmatrix} \frac{\partial f}{\partial x} x'_1 & \frac{\partial f}{\partial y} x'_1 \\ \frac{\partial g}{\partial x} y_1 & \frac{\partial g}{\partial y} y'_1 \end{vmatrix} = \begin{vmatrix} 2 - 2x - y & -x \\ -1 - 2y & 3 - 2x \end{vmatrix}$$

a.)
1. $x = 0, y = 0$: No fish at all (extinction of both species).
2. $x = \frac{\epsilon_1}{\sigma_1}, y = 0$: Only bluegill survive, and redear go extinct.
3. $x = 0, y = \frac{\epsilon_2}{\sigma_2}$: Only redear survive, and bluegill go extinct.

The inequalities are:

$$\frac{\epsilon_2}{\alpha_2} > \frac{\epsilon_1}{\sigma_1}, \quad \frac{\epsilon_2}{\sigma_2} > \frac{\epsilon_1}{\alpha_1}.$$

Interpretation of the Inequalities

- $\frac{\epsilon_2}{\alpha_2} > \frac{\epsilon_1}{\sigma_1}$: The redear population (y) has a stronger ability to outcompete bluegill (x) than bluegill's self-limiting density dependence.
- $\frac{\epsilon_2}{\sigma_2} > \frac{\epsilon_1}{\alpha_1}$: The redear population is less affected by its own density (y) than by competition from (x).

Conclusion => Bluegill is driven to extinct over time $x \to 0$. $y = \frac{\epsilon_2}{\sigma_2}$

6.)

Part (b):

The inequalities are:

$$\frac{\epsilon_1}{\sigma_1} > \frac{\epsilon_2}{\alpha_2}, \quad \frac{\epsilon_1}{\alpha_1} > \frac{\epsilon_2}{\sigma_2}.$$

Interpretation of the Inequalities

- $\frac{\epsilon_1}{\sigma_1} > \frac{\epsilon_2}{\alpha_2}$: The bluegill population (x) has a stronger ability to outcompete redear (y) than redear's self-limiting density dependence.
- $\frac{\epsilon_1}{\alpha_1} > \frac{\epsilon_2}{\sigma_2}$: The bluegill population is less affected by its own density (x) than by competition from redear (y).

↓ redear extinction overtime

Setting $x' = 0$, we get the equation for critical points involving x:

$$x\left(1 - 0.2x - \frac{2y}{x+6}\right) = 0.$$

Setting $y' = 0$, we get the equation for critical points involving y:

$$y\left(-0.25 + \frac{x}{x+6}\right) = 0.$$

From the first equation, we can see that either x = 0 or $1 - 0.2x - \frac{2y}{x+6} = 0$.

From the second equation, we can see that either y = 0 or $-0.25 + \frac{x}{x+6} = 0$.

Solving $-0.25 + \frac{x}{x+6} = 0$ gives $x = 3$.

So the critical points are:

1. (0, 0)
2. (3, 0)

Now, we need to find the corresponding y values for the critical points:

For (0, 0), we can use the first equation $1 - 0.2x - \frac{2y}{x+6} = 0$ to find y. Substituting $x = 0$, we get $1 - \frac{2y}{6} = 0$, which gives y = 3.

For (3, 0), using the same equation, we get $1 - 0.6 - \frac{2y}{9} = 0$, which gives $y = 2.25$.

(b) Stability Analysis:

Now we need to analyze the stability of these critical points. To do this, we can compute the Jacobian matrix:

$$J(x, y) = \begin{bmatrix} 1 - 0.2x - \frac{2y}{x+6} & -\frac{2x}{(x+6)^2} \\ \frac{y}{(x+6)^2} & -0.25 + \frac{x}{x+6} \end{bmatrix}$$

Evaluate J at each critical point and analyze the eigenvalues. If both eigenvalues have negative real parts, the critical point is a stable node. If both eigenvalues have positive real parts, the critical point is a saddle. If one eigenvalue is positive and the other is negative, it's a saddle as well, but unstable.

For (0, 3), the Jacobian matrix becomes: $J(0, 3) = \begin{bmatrix} 1 & 0 \\ 0 & -0.25 \end{bmatrix}$

Eigenvalues are 1 and -0.25, both real and one positive, one negative. Therefore, (0, 3) is a saddle.

For (3, 2.25), the Jacobian matrix becomes: $J(3, 2.25) = \begin{bmatrix} 0.4 & -0.08 \\ 0.0625 & 0 \end{bmatrix}$

Eigenvalues are 0.4 and 0, both non-negative. Therefore, (3, 2.25) is a saddle as well.

(c) Direction Field and Phase Portrait:

Given the stability analysis, both critical points are saddles. Below is an approximate direction field and phase portrait for the system. Please note that the arrows in the direction field indicate the direction in which the solutions move. The critical points (0, 3)and (3, 2.25) are represented as red dots.

Critical points are found by setting $x' = 0$ and $y' = 0$.

1. From $x' = x(1 - \sigma x - 0.5y) = 0$:

$$x = 0 \quad \text{or} \quad 1 - \sigma x - 0.5y = 0.$$

If $x \neq 0$, then:

$$\sigma x + 0.5y = 1. \qquad (1)$$

2. From $y' = y(-0.75 + 0.25x) = 0$:

$$y = 0 \quad \text{or} \quad -0.75 + 0.25x = 0.$$

If $y \neq 0$, then:

$$x = 3. \qquad (2)$$

Step 2: Solve for Critical Points

From Equation (2), $x = 3$. Substitute $x = 3$ into Equation (1):

$$\sigma(3) + 0.5y = 1 \implies 3\sigma + 0.5y = 1.$$

Solve for y:

$$y = \frac{1 - 3\sigma}{0.5} = 2(1 - 3\sigma). \qquad (3)$$

Thus, the critical points are:

1. $(x, y) = (0, 0)$,
2. $(x, y) = (3, 0)$,
3. $(x, y) = (3, 2(1 - 3\sigma))$ (exists only when $1 - 3\sigma > 0$, i.e., $\sigma < \frac{1}{3}$).

(d) Effect of Increasing σ

- As σ increases from 0:
  - The critical point $(3, 2(1 - 3\sigma))$ moves closer to $(3, 0)$.
  - At $\sigma = \frac{1}{3}$, the critical point $(3, 2(1 - 3\sigma))$ disappears.
- For $\sigma > \frac{1}{3}$, only $(0, 0)$ and $(3, 0)$ remain.

Step 2: Evaluate the Jacobian at Each Critical Point

1. **At $(0, 0)$:**

$$J(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & -0.75 \end{pmatrix}.$$

Eigenvalues: $\lambda_1 = 1, \lambda_2 = -0.75$.
- Since one eigenvalue is positive, $(0, 0)$ is an **unstable saddle point**.

2. **At $(3, 0)$:**

$$J(3, 0) = \begin{pmatrix} 1 - 2\sigma(3) & -0.5(3) \\ 0 & -0.75 + 0.25(3) \end{pmatrix} = \begin{pmatrix} 1 - 6\sigma & -1.5 \\ 0 & 0 \end{pmatrix}.$$

Eigenvalues: $\lambda_1 = 0, \lambda_2 = 1 - 6\sigma$.
- Stability depends on $1 - 6\sigma$:
  - If $\sigma < \frac{1}{6}$, $(3, 0)$ is **unstable**.
  - If $\sigma > \frac{1}{6}$, $(3, 0)$ is **stable**.

3. **At $(3, 2(1 - 3\sigma))$:** This critical point exists only for $\sigma < \frac{1}{3}$. Substitute $x = 3, y = 2(1 - 3\sigma)$ into $J(x, y)$:

$$J(3, 2(1 - 3\sigma)) = \begin{pmatrix} 1 - 6\sigma - (1 - 3\sigma) & -1.5 \\ 0.5(1 - 3\sigma) & -0.75 + 0.75 \end{pmatrix}.$$

Simplify:

$$J(3, 2(1 - 3\sigma)) = \begin{pmatrix} -5\sigma & -1.5 \\ 0.5(1 - 3\sigma) & 0 \end{pmatrix}.$$

Behavior Change: Find Crit P.n.t.

(c.) In term of alpha → $x' = \begin{pmatrix} -1 & -1 \\ -\alpha & -1 \end{pmatrix}x$ $\quad 1 - (-\alpha \times -1) = 1 - \alpha$

$\lambda^2 + 2\lambda + (1 - \alpha) = 0$

$\frac{-2 \pm \sqrt{4 - 4(1)(1-\alpha)}}{2}$

$= \frac{-2 \pm \sqrt{-4 + 4 + 4\alpha}}{2}$

$\lambda = -1 \pm \sqrt{\alpha}$

between 0.5 to 1 it is still $\lambda_1, \lambda_2 < 0$ → Nodal sink (Stable)

$-1 + 1 = 0$ → Positive
$-1 - 1 = -2$ → negative

as alpha approach to 1 it changes from Nodal sink to saddle as in case

If both Eigen value have positive real part then it is unstable

Definition: Piecewise continuous functions

A function f is called **piecewise continuous over the interval** $I \subset \mathbb{R}$ if f is defined over I and continuous at every point of I except for a finite set of points $t_1, \dots, t_N \in I$, at which f has finite left and right limits.

Example

$$f(t) := \begin{cases} 0, & \text{if } t < 0 \\ 1, & \text{if } t \geq 0. \end{cases}$$

Function f is discontinuous, and we can prove that there exists no differentiable function x such that $x'(t) = f(t)$ for all $t \in \mathbb{R}$. However, there are situations in which we may want to interpret and solve such an equation. Ideas?

Tranform table

Definition: Improper integral

Let f be a function defined on an unbounded interval $[a, \infty)$. If the integral $\int_a^T f(t)dt$ is defined for all $T > a$ and admits a finite limit as $T \to \infty$, then we say that the **improper integral** of f over $[a, \infty)$ is **convergent** and write:

$$\int_a^\infty f(t)dt := \lim_{T\to\infty} \int_a^T f(t)dt.$$

If the limit is infinite or does not exist, then the improper integral $\int_a^\infty f(t)dt$ **diverges**.

Remark

- We can have a convergent $\int_a^\infty f(t)dt$ while $f(t)$ does not converge to 0.
- If $\int_a^\infty f(t)dt$ converges, then so does the series $\sum a_n$, with $a_n := \int_n^{n+1} f(t)dt$.
- The converse is not true: take $f(t) = \sin(2\pi t)$.
- If $f(t) \geq 0$ for all $t \geq a$, then if for some M, then $T \mapsto \int_a^T f(t)dt$ is an increasing function of T, and thus has a limit $L \in \mathbb{R} \cup \{\infty\}$ as $T \to \infty$. (We still say that

$L(1) = \frac{1}{s}$

$L(e^{at}) = \frac{1}{s-a}$

$L(t^n) = \frac{n!}{s^{n+1}}$

$L(\sin(at)) = \frac{a}{s^2+a^2}$

$L(\cos(at)) = \frac{s}{s^2+a^2}$

$L(f'(t)) = sF(s) - f(0)$

$L(f''(t)) = s^2F(s) - sF(0) - f'(0)$

$L(\cosh(at))(s) = \frac{s}{s^2-a^2}, \quad s > |a|$

$L(\sinh(at))(s) = \frac{a}{s^2-a^2}, \quad s > |a|$

→ Table of transform

Exercise

Find the Laplace transform of $\int_0^\infty e^{-st} f(t)dt$

$$f(t) = \begin{cases} \frac{\sin(t)}{t}, & t > 0, \\ 0, & t = 0. \end{cases} \Rightarrow \boxed{\mathcal{L}(f)(s) = \frac{\pi}{2} - \arctan(s)}, \quad s > 0$$

$F(s) := \mathcal{L}(f)(s) = \int_0^\infty e^{-st} \frac{\sin(t)}{t} dt$

$F'(s) = \frac{d}{ds}\left(\int_0^\infty e^{-st} \frac{\sin t}{t} dt\right) = \int_0^\infty \frac{d}{ds}(e^{-st}) \frac{\sin(t)}{t} dt = \int_0^\infty \frac{-t}{t} e^{-st} \sin(t) dt$

$= -\mathcal{L}(\sin)(s) = \frac{-1}{s^2+1^2}, \quad s > 0$

$F'(s) = -\frac{1}{s^2+1} \Rightarrow F(s) = -\int \frac{1}{1+s^2} ds = C - \arctan(s)$

$F(s) \xrightarrow[s\to\infty]{} 0$ because f is bounded. So: $C = \lim_{s\to\infty} \arctan(s) = \frac{\pi}{2}$.

Theorem: Comparison

Let f, g be piecewise continuous functions defined over $[a, \infty)$ satisfying, for all t large enough, $0 \leq f(t) \leq g(t)$. Then:

$$\int_a^\infty g(t)dt < \infty \Rightarrow \int_a^\infty f(t)dt < \infty$$

$$\int_a^\infty f(t)dt = \infty \Rightarrow \int_a^\infty g(t)dt = \infty.$$

Definition: Absolute convergence

If $\int_a^\infty |f(t)|dt < \infty$, then $\int_a^\infty f(t)dt$ is convergent. We say that $\int_a^\infty f(t)dt$ is **absolutely convergent**.

Remark

A convergent improper integral is not necessarily absolutely convergent: *e.g.*, $\int_a^\infty \frac{\sin t}{t}dt$.

- In all the applications, we assume that the solution y of the studied ODE satisfies the hypotheses that allow us to take its Laplace transform: y is piecewise continuous and of exponential order, and so are all of its derivatives involved in the ODE.
- When we find y, we need to either check that y solves the ODE...
- ... or check that y satisfies the conditions that justify the use of the Laplace transform.

6.2 Solution of initial Value problem

Inverse table transformation.

$\mathbf{f(t) = \mathcal{L}^{-1}(F(s))}$	$\mathbf{F(s) = \mathcal{L}(f(t))}$
$e^{at}t^n, \quad n \in \mathbb{N}$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
$e^{at}f(t)$	$F(s-a)$
$cf(ct)$	$F\left(\frac{s}{c}\right)$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$
f'	$sF(s) - f(0)$
f''	$s^2F(s) - sf(0) - f'(0)$
$f^{(n)}$	$s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{(k-1)}(0)$
$(-t)^nf$	$F^{(n)}(s)$

2 improper integral to remember

1.) $\int_0^\infty e^{-ct}dt \Rightarrow \begin{cases} \frac{1}{c} & \text{if } c > 0 \\ \infty, & \text{if } c \leq 0 \end{cases}$

$= \lim_{T\to\infty} \left[-\frac{1}{c}e^{-ct}\right]_0^T \Rightarrow \lim_{T\to\infty} \frac{1}{c}[1 - e^{-cT}] = \frac{1}{c}$ for $c > 0$

if $c \leq 0 \to +\infty$

2.) $\int_0^\infty t^s dt = \begin{cases} \frac{1}{s+1}, & \text{if } s < -1 \\ \infty, & \text{if } s \geq -1. \end{cases}$

$\lim_{T\to\infty} \left[\frac{t^{s+1}}{s+1}\right]_0^T = \frac{T^{s+1} - 0^{s+1}}{s+1}$

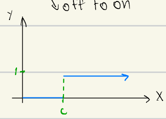
$\xrightarrow[T\to\infty]{} \infty$ if $s+1 > 0$

$\xrightarrow[T\to\infty]{} \frac{1}{s+1}$ if $s+1 < 0$

Step function. (Heaviside function)

$u_c(t) = u(t-c) = \begin{cases} 0 & 0 \leq t < c \\ 1 & t \geq c \end{cases}$ off to on

or $H(t)$ $H(t-c)$



Formula

$hu_c(t) = \begin{cases} 0 \\ h \end{cases}$

$1 - u_c(t) = \begin{cases} 1 \\ 0 \end{cases}$

$h(1 - u_c(t)) = \begin{cases} h \\ 0 \end{cases}$

on to off

If $c = 3$ (switch occur at $c = 3$)

then → $u_3(t) = u(t-3) = \begin{cases} 0 & 0 \leq t < 3 \\ 1 & t \geq 3 \end{cases}$

Can be write either one of this

Ex. $f(t) = \begin{cases} -2 & 0 \leq t < 1 \\ 4 & 1 \leq t < 3 \\ 8 & 3 \leq t < 5 \\ 12 & t \geq 5 \end{cases}$ → $t=1$ $u_1(t)$; $t=3$ $u_3(t)$; $t=5$ $u_5(t)$ — break pnt.

$f(t) = -2 + 6u_1(t) + 4u_3(t) + 4u_5(t)$

Can multiply by constant → $f(t)u_c(t) = \begin{cases} 0 \\ f(t) \end{cases}$

$L(u(t-c)) = \frac{e^{-cs}}{s}$

$L(f(t-c)u(t-c)) = e^{-cs}F(s)$

$L(g(t)u(t-c)) = e^{-cs}L(g(t+c))$

Definition: Laplace Transform

Let f be a function such that the integral below is defined for some $s \in \mathbb{R}$. The **Laplace transform** of $f(t)$ is function $F(s)$, defined as

$$\mathcal{L}(f)(s) = F(s) := \int_0^\infty e^{-st}f(t)dt \quad (1)$$

always assume $a - s < 0$

for all s for which the improper integral converges. We denote: $F = \mathcal{L}(f)$.

Note that $\mathcal{L}(f)$ only depends on the values of $f(t)$ for $t \geq 0$.

Exercise

Express function f as a linear combination of Heaviside functions, and compute its Laplace transform:

$$f(t) = \begin{cases} 1, & t < 3, \\ -3, & 3 \leq t < 4, \\ 2, & 4 \leq t < 6, \\ 4, & 6 \leq t < 7. \end{cases}$$

$f(t) = 1 - 4u_3(t) + 5u_4(t) + 2u_6(t)$

$$\mathcal{L}(u_0)(s) = \mathcal{L}(1)(s) = \frac{1}{s}, \quad s > 0,$$

$$\mathcal{L}(u_c)(s) = \mathcal{L}(u_0(t-c))(s) = \frac{e^{-cs}}{s}, \quad s > 0.$$

Step function table

Laplace transform for step function

Ex. $y'' + 4y' = (\sin t)u(t - 2\pi)$; $y(0) = 1$; $y'(0) = 0$

let $g(t) = \sin t$, $c = 2\pi$

$g(t + 2\pi) = \sin(t + 2\pi)$ — $\sin(t+2\pi) = \sin(t)$

Transform: $s^2Y(s) - sy(0) - y'(0) + 4(sY(s) - y(0)) = e^{-2\pi s}L(\sin(t))$

$= s^2Y(s) - s + 4sY(s) - 4 = e^{-2\pi s} + \frac{1}{s^2+1}$

$Y(s) = \frac{1}{s} + \frac{1}{s(s^2+1)(s+4)}e^{-2\pi s}$

$= \frac{1}{s} + \left(\frac{A}{s} + \frac{B}{s+4} + \frac{Cs+D}{s^2+1}\right)e^{-2\pi s}$

Convolution

Theorem

Let f be a function defined over $\mathbb{R}$ such that:

- f is piecewise continuous over every interval $[0, T]$, $T > 0$,
- f is of exponential order: there exist constants $K > 0$, $M > 0$ and $a \in \mathbb{R}$ such that
$$\forall t \geq M, \quad |f(t)| \leq Ke^{at}.$$

Then the Laplace transform $F := \mathcal{L}(f)$ is defined for all $s > a$.

Definition

Let g and h be piece-wise continuous functions defined for $t \geq 0$. We define their convolution f for $t \geq 0$ as

$$f(t) := (g * h)(t) = \int_0^t g(\tau)h(t-\tau)d\tau, \quad t \geq 0.$$

Impulse Function

The Dirac delta δ_a admits a Laplace transform, and for $a > 0$, we have

$$\mathcal{L}(\delta_a) = \int_0^\infty e^{-st}\delta_a(t)dt = e^{-sa}.$$

The case $a = 0$ is more delicate, since the Laplace transform is defined on $\mathbb{R}_+$.

Example

We have

$$\mathcal{L}\left(\sum \alpha_i\delta_{a_i}\right)(s) = \sum \alpha_i e^{-a_is}.$$

Solve the IVP

$$y'' + 3y' + 2y = \delta_1 - 2\delta_2, \qquad y(0) = 1; \; y'(0) = -1.$$

Let $Y = \mathcal{L}(y)$. Then:

$$\mathcal{L}(y') = sY(s) - y(0) = sY(s) - 1,$$
$$\mathcal{L}(y'') = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s + 1.$$

We get:

$$(s^2 + 3s + 2)\,Y(s) = e^{-s} - 2e^{-2s} + s + 2,$$

hence:

$$Y(s) = \frac{e^{-s}}{s^2+3s+2} - \frac{2e^{-2s}}{s^2+3s+2} + \frac{s+2}{s^2+3s+2}.$$

We have: $\frac{1}{s^2+3s+2} = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$, so:

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+3s+2}\right) = \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+2}\right) = e^{-t} - e^{-2t}.$$

Hence:

$$\mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2+3s+2}\right) = u_1(t)\left(e^{-(t-1)} - e^{-2(t-1)}\right),$$
$$\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s^2+3s+2}\right) = u_2(t)\left(e^{-(t-2)} - e^{-2(t-2)}\right),$$
$$\mathcal{L}^{-1}\left(\frac{s+2}{s^2+3s+2}\right) = \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) = e^{-t}.$$

Laplace transform by Integration

$L(tf(t)) = -\frac{d}{ds}L(f(t))\,ds$

$L(t^nf(t)) = \frac{d^n}{d^ns}L(f(t))\,ds$ — not the same as $f(t)$!! (the negative)

$\cos at \Rightarrow \frac{e^{iat} + e^{-iat}}{2}$

In each of Problems 1 through 12:
(a) Find the solution of the given initial value problem.
(b) Draw a graph of the solution.

1. $y'' + 2y' + 2y = \delta(t - \pi)$; $y(0) = 1$, $y'(0) = 0$

1.) $y'' + 2y' + 2y = \delta(t - \pi)$; $y'' + 2y' + 2y = e^{-\pi s}$

$s^2Y(s) - sy(0) - y'(0) + 2sY(s) - 2y(0) + 2Y(s) = e^{-\pi s}$

$s^2Y(s) - s + 2sY(s) - 2 + 2Y(s) = e^{-\pi s}$

$Y(s)(s^2 + 2s + 2) = e^{-\pi s} + s + 2$

$Y(s) = \frac{2}{(s+1)^2+1} + \frac{s}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}e^{-\pi s}$

Manipulate to get $s+1$ at the top.

$Y(s) = \frac{2}{(s+1)^2+1} + \frac{(s+1)-1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}(e^{-\pi s})$

$Y(s) = \frac{2}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}(e^{-\pi s})$

$Y(s) = \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}(e^{-\pi s})$

$y(t) = e^{-t}\sin t + e^{-t}\cos(t) + (e^{-(t-\pi)}\sin(t - \pi))u_\pi(t)$

Example $h(t-\tau) = t - \tau$

Let $g(t) = t^2$ and $h(t) = t$. We have:

$$g * h(t) = \int_0^t g(\tau)h(t-\tau)d\tau = \int_0^t \tau^2(t-\tau)d\tau$$

$$= t\int_0^t \tau^2 d\tau - \int_0^t \tau^3 d\tau = \frac{t^4}{3} - \frac{t^4}{4} = \frac{t^4}{12}.$$

10.) $F(s) = \frac{2s-3}{s^2+2s+10}$

$= \frac{2s-3}{(s+1)^2+9} = \frac{2s+2-5}{(s+1)^2+9} = \frac{2(s+1)}{(s+1)^2+9} - \frac{5}{(s+1)^2+9}$

$a = -1$, $b = 3$

$2e^{-t}\cos(3t) - \frac{5}{3}e^{-t}\sin(3t)$

29. Suppose that f and f' are continuous for $t \geq 0$ and of exponential order as $t \to \infty$. Use integration by parts to show that if $F(s) = \mathcal{L}\{f(t)\}$, then $\lim_{s\to\infty} F(s) = 0$. The result is actually true under less restrictive conditions, such as those of Theorem 6.1.2.

$F(s) = \mathcal{L}\{f(t)\}$

$F(s) = \int_0^\infty e^{-st}f(t)\,dt$

let $u = f(t)$, $dv = e^{-st}dt$

$F(s) = \frac{1}{s}f(0) + \int_0^\infty \frac{1}{s}e^{-st}f'(t)$

$\lim_{s\to\infty} F(s) = 0 + \lim_{s\to\infty} \int_0^\infty \frac{1}{s}e^{-st}|f'(t)|\,dt$

Comparison theorem: We know $f'(t) \leq Ke^{at}$ (exponential order)

Hence $\lim_{s\to\infty} F(s) = 0 + \lim_{s\to\infty} \int_0^\infty \frac{1}{s}e^{-st}(Ke^{at})dt$

$\lim_{s\to\infty} F(s) \to \lim_{s\to\infty} \frac{K}{s}\int_0^\infty e^{-st}e^{at}dt \Rightarrow \frac{K}{s}\int_0^\infty e^{(a-s)t}dt$

$\lim_{s\to\infty} \frac{K}{s}\left(\frac{1}{a-s}\right)\left[e^{(a-s)t}\right]_0^\infty$ — $a - s < 0$

$\lim_{s\to\infty} \frac{K}{s}\left(\frac{1}{a-s}\right)[0 - 1]$

$\lim_{s\to\infty} = 0$

Leading to comparison theorem, As $\lim_{s\to\infty} \int_0^\infty \frac{1}{s}e^{-st}(Ke^{at}) = 0$ and $Ke^{at} > e^{-st}f'(t)$. Hence $\lim_{s\to\infty} F(s)$ also $= 0$

$F(s) = \mathcal{L}(f)(s) = \int_0^\infty e^{-st}f(t)\,dt.$

Find the Laplace transform of t^ne^{at}. It's important to assume that $a - s < 0$, that is, $s > a$.

$$\mathcal{L}(t^ne^{at}) = \int_0^\infty e^{-st}t^ne^{at}dt = \int_0^\infty t^ne^{(a-s)t}dt = \dots = \frac{n!}{(s-a)^{n+1}}$$

30. **The Gamma Function.** The gamma function is denoted by $\Gamma(p)$ and is defined by the integral

$$\Gamma(p+1) = \int_0^\infty e^{-x}x^p dx.$$

The integral converges as $x \to \infty$ for all p. For $p < 0$ it is also improper at $x = 0$, because the integrand becomes unbounded as $x \to 0$. However, the integral can be shown to converge at $x = 0$ for $p > -1$.
(a) Show that, for $p > 0$,

$$\Gamma(p+1) = p\Gamma(p).$$

(b) Show that $\Gamma(1) = 1$.

(a) $\int_0^\infty e^{-x}x^p dx \Rightarrow u = x^p$, $dv = e^{-x}$; $v = -e^{-x}$

$-e^{-x}x^p\Big|_0^\infty + \int_0^\infty e^{-x}px^{p-1}dx$

$\Gamma(p+1) = 0 + p\int_0^\infty e^{-x}x^{p-1}dx$

Is the form $\Gamma(p)$

Hence $\Gamma(p+1) \Rightarrow p\,\Gamma(p)$

$$Y(s) = \frac{s^2 - 4s + 7}{s^3 - 4s^2 + 6s^2 - 4s + 1} = \frac{s^2 - 4s + 7}{(s-1)^4} = \frac{1}{(s-1)^2} - \frac{2}{(s-1)^3} + \frac{4}{(s-1)^4}$$

Take the inverse Laplace transform of $Y(s)$ now to recover $y(t)$.

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2} - \frac{2}{(s-1)^3} + \frac{4}{(s-1)^4}\right\} = te^t - t^2e^t + \frac{2}{3}t^3e^t = \frac{1}{3}te^t(3 - 3t + 2t^2)$$

26. $y'' + 4y = \begin{cases} t, & 0 \le t < 1, \\ 1, & 1 \le t < \infty; \end{cases} \quad y(0) = 0, \quad y'(0) = 0$

Convert

$L(s) = s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \int_0^1 e^{-st}(t)\,dt + \int_1^\infty e^{-st}\,dt$

$\frac{1}{s^2} - \frac{e^{-s}}{s} - \frac{e^s}{s^2} \qquad \frac{e^{-s}}{s}$

Both sides of the Taylor series expansion for sin t.

$\mathcal{L}\{\sin t\} = \mathcal{L}\left\{\sum_{n=0}^\infty \frac{(-1)^n t^{2n+1}}{(2n+1)!}\right\} = \sum_{n=0}^\infty \mathcal{L}\left\{\frac{(-1)^n t^{2n+1}}{(2n+1)!}\right\} = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!}\mathcal{L}\{t^{2n+1}\} = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!}\cdot\frac{(2n+1)!}{s^{(2n+1)+1}} = \sum_{n=0}^\infty \frac{(-1)^n}{s^{2n+2}} = \frac{1}{s^2}\sum_{n=0}^\infty \frac{(-1)^n}{s^{2n}} = \frac{1}{s^2}\sum_{n=0}^\infty\left(-\frac{1}{s^2}\right)^n = \frac{1}{s^2}\frac{1}{1-\left(-\frac{1}{s^2}\right)} = \frac{1}{s^2+1}$

(a) Show that $F'(s) = \mathcal{L}\{-tf(t)\}$.

(b) Show that $F^{(n)}(s) = \mathcal{L}\{(-t)^n f(t)\}$; hence differentiating the Laplace transform corresponds to multiplying the original function by $-t$.

Differentiate both sides with respect to s.

$F'(s) = \frac{d}{ds}\int_0^\infty e^{-st}f(t)\,dt = \int_0^\infty \frac{\partial}{\partial s}e^{-st}f(t)\,dt = \int_0^\infty(-te^{-st})f(t)\,dt = \int_0^\infty e^{-st}[-tf(t)]\,dt = \mathcal{L}\{-tf(t)\}$

Part (b)

Differentiate both sides n times with respect to s.

$F^{(n)}(s) = \frac{d^n}{ds^n}\int_0^\infty e^{-st}f(t)\,dt = \int_0^\infty \frac{\partial^n}{\partial s^n}e^{-st}f(t)\,dt = \int_0^\infty[(-t)^n e^{-st}]f(t)\,dt = \int_0^\infty e^{-st}[(-t)^n f(t)]\,dt = \mathcal{L}\{(-t)^n f(t)\}$

Integrating factor for first ODE - $s^{(6c)}$

$Y'(s) + s^2 Y(s) = s$ $P(s)$, $f(x)$

$I = e^{\int P(s)ds} = e^{\int s^2 ds} = e^{\frac{s^3}{3}}$

Multiply the whole integral by I =>

$e^{\frac{s^3}{3}}(Y'(s)) + e^{\frac{s^3}{3}}s^2(Y(s)) = se^{\frac{s^3}{3}}$

$\frac{d}{ds}\left(Y(s)e^{\frac{s^3}{3}}\right) = se^{\frac{s^3}{3}}$

$e^{\frac{s^3}{3}}Y(s) = \int re^{\frac{r^3}{3}}\,dr + C_1$

Hence transform => $Y(s) = e^{\frac{-s^3}{3}}\left(\int_0^s re^{\frac{r^3}{3}}dr + C_1\right)$

(b) Show that $Y(s) = c(1+s^2)^{-1/2}$, where c is an arbitrary constant.

from question a. $(s^2+1)Y'(s) + sY(s) = 0$

$(s^2+1)Y'(s) = -sY(s)$

$\frac{Y'(s)}{Y(s)} = -\frac{s}{s^2+1}$

$\ln|Y(s)| = -\frac{1}{2}\ln(s^2+1) + C_1$

$e^{\ln|Y(s)|} = e^{\ln(s^2+1)^{-\frac{1}{2}} + C_1}$

$Y(s) = C_1(1+s^2)^{-\frac{1}{2}}$

36. Consider Bessel's equation of order zero

$ty'' + y' + ty = 0.$

Recall from Section 5.7 that $t = 0$ is a regular singular point for this equation, and therefore solutions may become unbounded as $t \to 0$. However, let us try to determine whether there are any solutions that remain finite at $t = 0$ and have finite derivatives there. Assuming that there is such a solution $y = \phi(t)$, let $Y(s) = \mathcal{L}\{\phi(t)\}$.

(a) Show that $Y(s)$ satisfies

$(1+s^2)Y'(s) + sY(s) = 0.$

(b) Show that $Y(s) = c(1+s^2)^{-1/2}$, where c is an arbitrary constant.

(c) Writing $(1+s^2)^{-1/2} = s^{-1}(1+s^{-2})^{-1/2}$, expanding in a binomial series valid for $s > 1$, and assuming that it is permissible to take the inverse transform term by term, show that

$y = c\sum_{n=0}^\infty \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2} = cJ_0(t),$

where J_0 is the Bessel function of the first kind of order zero. Note that $J_0(0) = 1$ and that J_0 has finite derivatives of all orders at $t = 0$. It was shown in Section 5.7 that the second solution of this equation becomes unbounded as $t \to 0$.

(a) As $Y(s) = L\{\phi(t)\}$ and $ty'' + y' + ty = 0$ are linear; then we can apply Laplace transform

$L(y(s)) = L(ty'') + L(y') + L(ty) = 0$

$L(tf(t)) = -\frac{d}{ds}L(f(t))ds = -\frac{d}{ds}L(y'') + L(y') - \frac{d}{ds}L(y)$

$=> -\frac{d}{ds}\left[s^2Y(s) - sy(0) - y'(0)\right] + sY(s) - y(0) - \frac{d}{ds}(Y(s)) = 0$

$= -2sY(s) - s^2Y'(s) + y(0) + sY(s) - y(0) - Y'(s) = 0$

$-sY(s)[-s^2-1]Y'(s) =$

Hence Prove => $(s^2+1)Y'(s) + sY(s)$

Transform of $t^2 \sin 6t$. !!!

$F(s) = \int_0^\infty e^{-st}t^2\sin bt\,dt = \int_0^\infty\left(\frac{\partial^2}{\partial s^2}e^{-st}\right)\sin bt\,dt = \frac{d^2}{ds^2}\int_0^\infty e^{-st}\sin bt\,dt = \frac{d^2}{ds^2}\mathcal{L}\{\sin bt\} = \frac{d^2}{ds^2}\left(\frac{b}{s^2+b^2}\right) = \frac{d}{ds}\left[-\frac{2bs}{(s^2+b^2)^2}\right] = -\frac{(2b)(s^2+b^2)^2 - 2bs[2(s^2+b^2)2s]}{(s^2+b^2)^4} = -\frac{2b(s^4+2s^2b^2+b^4) - 8bs^2(s^2+b^2)}{(s^2+b^2)^4} = -\frac{2b^5 - 4b^3s^2 - 6bs^4}{(s^2+b^2)^4} = \frac{2b(b^2-3s^2)(s^2+b^2)}{(s^2+b^2)^4} = \frac{2b(3s^2-b^2)}{(s^2+b^2)^3}$

$L(f(ct)) = \frac{1}{c}F\left(\frac{s}{c}\right)$

Part (a)

Suppose we use $f(ct)$ rather than $f(t)$.

$\mathcal{L}\{f(ct)\} = \int_0^\infty e^{-st}f(ct)\,dt.$

Make the substitution $x = ct$. Then $dx = c\,dt$.

$\mathcal{L}\{f(ct)\} = \int_0^\infty e^{-sx/c}f(x)\left(\frac{dx}{c}\right) = \frac{1}{c}\int_0^\infty e^{-(s/c)x}f(x)\,dx = \frac{1}{c}F\left(\frac{s}{c}\right)$

(b) Show that if k is a positive constant, then

$\mathcal{L}^{-1}\{F(ks)\} = \frac{1}{k}f\left(\frac{t}{k}\right).$

(c) Show that if a and b are constants with $a > 0$, then

$\mathcal{L}^{-1}\{F(as+b)\} = \frac{1}{a}e^{-bt/a}f\left(\frac{t}{a}\right).$

$t^n e^{at}$

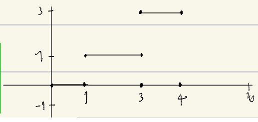
$F(s) = \int_0^\infty e^{-st}t^n e^{at}\,dt = \int_0^\infty\left[(-1)^n\frac{\partial^n}{\partial s^n}e^{-st}\right]e^{at}\,dt = (-1)^n\frac{d^n}{ds^n}\int_0^\infty e^{-st}e^{at}\,dt = (-1)^n\frac{d^n}{ds^n}\int_0^\infty e^{(a-s)t}\,dt = (-1)^n\frac{d^n}{ds^n}\left[\frac{1}{a-s}e^{(a-s)t}\right]_0^\infty = (-1)^n\frac{d^n}{ds^n}\left(\frac{1}{s-a}\right) = (-1)^n\frac{d^n}{ds^n}[(s-a)^{-1}] = (-1)^n[(-1)^n n!(s-a)^{-1-n}] = (-1)^{2n}\frac{n!}{(s-a)^{n+1}} = \frac{n!}{(s-a)^{n+1}}$

In each of Problems 1 through 6, sketch the graph of the given function on the interval $t \ge 0$.

1. $g(t) = u_1(t) + 2u_3(t) - 6u_4(t)$
2. $g(t) = (t-3)u_2(t) - (t-2)u_3(t)$
3. $g(t) = f(t-\pi)u_\pi(t)$, where $f(t) = t^2$
4. $g(t) = f(t-3)u_3(t)$, where $f(t) = \sin t$
5. $g(t) = f(t-1)u_2(t)$, where $f(t) = 2t$
6. $g(t) = (t-1)u_1(t) - 2(t-2)u_2(t) + (t-3)u_3(t)$

1.) $g(t) = u_1(t) + 2u_3(t) - 6u_4(t)$

$g(t) = H(t-1) + 2H(t-3) - 6H(t-4)$



Laplace for $t - u_1(t)(t-1)$

$F(s) = \int_0^\infty e^{-st}[t-(t-1)H(t-1)]\,dt = \int_0^\infty te^{-st}\,dt - \int_0^\infty(t-1)e^{-st}H(t-1)\,dt = \int_0^\infty te^{-st}\,dt - \int_1^\infty(t-1)e^{-st}\,dt = \int_0^\infty te^{-st}\,dt - \int_1^\infty te^{-st}\,dt + \int_1^\infty e^{-st}\,dt = \int_0^\infty\left(-\frac{\partial}{\partial s}e^{-st}\right)dt - \int_1^\infty\left(-\frac{\partial}{\partial s}e^{-st}\right)dt + \int_1^\infty e^{-st}\,dt$

38. Suppose that

$g(t) = \int_0^t f(\tau)\,d\tau.$

If $G(s)$ and $F(s)$ are the Laplace transforms of $g(t)$ and $f(t)$, respectively, show that

$G(s) = F(s)/s.$

$G(s) = L(g(t)) = \int_0^\infty e^{-st}\left[\int_0^t f(\tau)\,d\tau\right]dt$

Laplace transform → $L(g(t)) = \int_0^\infty e^{-st}$

$=> \int_0^\infty\int_\tau^\infty e^{-st}f(\tau)\,dt\,d\tau$

$=> \int_0^\infty\left(\frac{1}{s}e^{-st}\Big|\right)f(\tau)\,d\tau$

$\frac{1}{s}\int_0^\infty e^{-s\tau}f(\tau)\,d\tau$

$\frac{1}{s}L(f(t)) => \frac{F(s)}{s}$

14. Find an expression involving $u_c(t)$ for a function f that ramps up from zero at $t = t_0$ to the value h at $t = t_0 + k$.

15. Find an expression involving $u_c(t)$ for a function g that ramps up from zero at $t = t_0$ to the value h at $t = t_0 + k$ and then ramps back down to zero at $t = t_0 + 2k$.

$g(t) = \frac{h}{k}(t-t_0)H(t-t_0) - \frac{h}{k}(t-t_0)H(t-t_0-k) + \frac{h}{k}(t_0+2k-t)H(t-t_0-k) - \frac{h}{k}(t_0+2k-t)H(t-t_0-2k)$

$f(t) = \frac{h}{k}(t-t_0)H(t-t_0) - \frac{h}{k}(t-t_0)H(t-t_0-k) + hH(t-t_0-k)$

For the first term, the Heaviside function $H(t-t_0)$, defined to be 1 if $t > t_0$ and 0 if $t < t_0$, make the function start increasing at $t = t_0$. The factor of $(h/k)(t-t_0)$ makes it so that it reaches a height of h at $t = t_0 + k$. The second term, which activates when $t = t_0 + k$, cancels the first term. The third term makes $f(t)$ have a value of h from $t = t_0 + k$ and on. $f(t)$ can also be written as

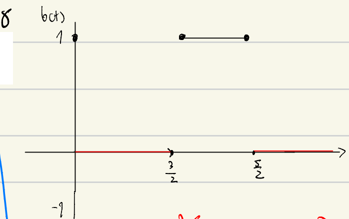
Spring Mass System

$u'' + \frac{1}{4}u' + u = kg(t), \quad u(0) = 0, \quad u'(0) = 0,$

$= \left\{k - ke^{-(t-\frac{3}{2})/8}\cos\left[\frac{\sqrt{63}}{8}\left(t-\frac{3}{2}\right)\right] - \frac{k}{\sqrt{63}}e^{-(t-\frac{3}{2})/8}\sin\left[\frac{\sqrt{63}}{8}\left(t-\frac{3}{2}\right)\right]\right\}H\left(t-\frac{3}{2}\right) - \left\{k - ke^{-(t-\frac{5}{2})/8}\cos\left[\frac{\sqrt{63}}{8}\left(t-\frac{5}{2}\right)\right] - \frac{k}{\sqrt{63}}e^{-(t-\frac{5}{2})/8}\sin\left[\frac{\sqrt{63}}{8}\left(t-\frac{5}{2}\right)\right]\right\}H\left(t-\frac{5}{2}\right)$

$A = 1$; $C = -\frac{1}{4}$; $B = -1$

$t = 25.68$

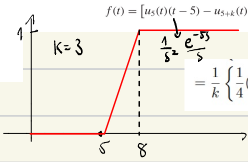


graph of $g(t)$

Modify the problem in Example 2 of this section by replacing the given forcing function $g(t)$ by

$f(t) = [u_5(t)(t-5) - u_{5+k}(t)(t-5-k)]/k.$

Solve initial Value Problem

(a.) 

$K = 3$

$\frac{1}{s^2}\frac{e^{-5s}}{s} \qquad \frac{1}{s^2}\frac{e^{(5-k)s}}{s}$

$= \frac{1}{k}\left\{\frac{1}{4}(t-5) - \frac{1}{8}\sin[2(t-5)]\right\}u_5(t) - \frac{1}{k}\left\{\frac{1}{4}(t-5-k) - \frac{1}{8}\sin[2(t-5-k)]\right\}u_{5+k}(t)$

Impulse Ex.

4. $y'' - y = -20\delta(t-3); \quad y(0) = 1, \quad y'(0) = 0$

$s^2Y(s) - sy(0) - y'(0) - Y(s) = -20e^{-3s}$

$s^2Y(s) - s - Y(s) = -20e^{-3s}$

$= Y(s)(s^2-1) = -20e^{-3s} + s$

$Y(s) = -20\left(\frac{1}{s^2-1}\right)e^{-3s} + \frac{s}{s^2-1}$

$Y(t) = -20\sinh(t-3)u_3(t) + \cosh(t)$

Consider the initial value problem

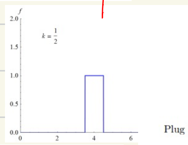
$y'' + \frac{1}{3}y' + 4y = f_k(t), \quad y(0) = 0, \quad y'(0) = 0,$

where

$f_k(t) = \begin{cases} 1/2k, & 4-k \le t < 4+k \\ 0, & 0 \le t < 4-k \quad \text{and} \quad t \ge 4+k \end{cases}$

graph

$f_k(t) => \frac{1}{2k}u_{(4-k)}(t) - \frac{1}{2k}u_{(4+k)}(t)$



Integrating factor

$y = \frac{1}{I(x)}\left[\int I(x)Q(x)\,dx + C\right]$ → from $y' + P(x)y = Q(x)$

Convolution Ex.

$f(t) = \int_0^t (t-\tau)^2\cos 2\tau\,d\tau$

Solution

Recognize that $f(t)$ is a convolution integral of the two functions, t and $\cos 2t$. The Laplace transform of $f(t)$ can be found by using the convolution theorem, which states that

$\mathcal{L}\left\{\int_0^t g(t-\tau)h(\tau)\,d\tau\right\} = G(s)H(s),$

where G and H are the Laplace transforms of g and h, respectively. Therefore,

$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{\int_0^t(t-\tau)^2\cos 2\tau\,d\tau\right\} = \mathcal{L}\{t^2\}\mathcal{L}\{\cos 2t\} = \left(\frac{2!}{s^{2+1}}\right)\left(\frac{s}{s^2+4}\right) = \frac{2s}{s^3(s^2+4)} = \frac{2}{s^2(s^2+4)}.$

$[s^2Y(s) - sy(0) - y'(0)] + Y(s) = \int_0^\infty e^{-st}u_0(t)\,dt + 2\sum_{k=1}^n(-1)^k\int_0^\infty e^{-st}u_{k\pi}(t)\,dt$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 0$.

$[s^2Y(s)] + Y(s) = \int_0^\infty e^{-st}\,dt + 2\sum_{k=1}^n(-1)^k\int_{k\pi}^\infty e^{-st}\,dt$

$(s^2+1)Y(s) = \frac{1}{s} + 2\sum_{k=1}^n(-1)^k\left(\frac{e^{-k\pi s}}{s}\right)$

Solve for $Y(s)$.

$Y(s) = \frac{1}{s(s^2+1)} + 2\sum_{k=1}^n(-1)^k\left[\frac{1}{s(s^2+1)}\right]e^{-k\pi s}$

Now write it in terms of known transforms by using partial fraction decomposition.

$Y(s) = \left(\frac{1}{s} - \frac{s}{s^2+1}\right) + 2\sum_{k=1}^n(-1)^k\left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-k\pi s}$

$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\left(\frac{1}{s} - \frac{s}{s^2+1}\right) + 2\sum_{k=1}^n(-1)^k\left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-k\pi s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+1}\right\} + 2\sum_{k=1}^n(-1)^k\mathcal{L}^{-1}\left\{\left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-k\pi s}\right\} = 1 - \cos t + 2\sum_{k=1}^n(-1)^k[1-\cos(t-k\pi)]H(t-k\pi) = 1 - \cos t + 2\sum_{k=1}^n(-1)^k[1-\cos(t-k\pi)]u_{k\pi}(t)$

Seperable

Ex. Find $\frac{dy}{dx} = \frac{\sec^2 x}{\tan^2 y}$

$\tan^2 y\,dy = \sec^2 x\,dx$

$\int \tan^2 y\,dy \int \sec^2 x\,dx$

$\int \sec^2 x - 1 \quad \tan x$

$\tan(y) - y = \tan x + C$

Note: * $y' = e^{x-y} => y' = (e^x)(e^{-y})$

$= \frac{1}{e^{-y}}dy = e^x\,dx$

Exact

$(2x-1)dx + (3y+7)dy = 0$

$\frac{\partial M}{\partial y} = 0, \quad \frac{\partial N}{\partial x} = 0$) exact

Hence $\int M dx = \frac{2x^2}{2} - x + g(y)$

$\int N dy = \frac{3y^2}{2} + 7y + f(x)$

Hence $C = x^2 - x + \frac{3}{2}y^2 + 7y$ Ans

Second Order ODE

$y'' + 9y = 0 \qquad m = 3i \quad = 0 \pm 3i$

$M^2 + 9 = 0$

$M^2 = -9$

$M = \sqrt{-9}$

$y = C_1\cos(3x) + C_2\sin(3x)$

$\mathcal{L}\{t^2\sin t\}$ $= \int_0^\infty e^{-st}t^2\sin(t)\,dt$

we want the case when $s = 1$, so plug $s = 1$ into answer from part (a)

$\frac{6 \cdot 1^2 - 2}{(1^2+1)^3}$

$= \frac{4}{8}$

$= \frac{1}{2}$

Linearisation.

$x'(t) = x(2-x-y)$

$y'(t) = -x + 3y - 2xy$

at $x = 0 \to y = 0$

as $2 - x - y = 0$

$y = 2 - x$

Sub to eq 2: $-x + 3(2-x) - 2x(2-x) = 0$

$-x + 6 - 3x - 4x + 2x^2 = 0$

$2x^2 - 8x + 6 = 0$

$x^2 - 4x + 3 = 0$

$(x-3)(x-1) = 0$

$x = 3, \; x = 1$

at $x = 3 \to y = -1$, at $x = 1 \to y = 1$

Crit Point. => $(0,0) \quad (3,-1) \quad (1,1)$

linearisation. →

$\begin{vmatrix} \frac{\partial f}{\partial x}x'_1 & \frac{\partial f}{\partial y}x'_1 \\ \frac{\partial g}{\partial x}y_1 & \frac{\partial g}{\partial y}y'_1 \end{vmatrix} = \begin{vmatrix} 2-2x-y & -x \\ -1-2y & 3-2x \end{vmatrix}$ $-1-(3)$

$x(1,1) = \begin{vmatrix} -1 & -1 \\ -3 & 1 \end{vmatrix} \quad \lambda^2 - 4 = 0$

$\lambda^2 = 4 \to \lambda = \pm 2$

as $\lambda_1 < 0$ and $\lambda_2 > 0$ it is Unstable

Unit Step function

If $f(t) = \begin{cases} 4t & \text{if } t < 2, \\ t^2+4 & \text{if } 2 \le t, \end{cases}$

$f(t) => 4t + (t^2 - 4t + 4)U_2(t)$

or $4t + (u(t-2))(t-2)^2$

$\frac{v''}{v'} = \frac{1}{2t}$

$\frac{d}{dt}(\ln(v'(t))) = \frac{1}{2t}$

$\ln(v'(t)) = \int\frac{1}{2t}dt$

$\ln|v'(t)| = \frac{1}{2}\ln(t) + C$

$|v'(t)| = e^{\frac{1}{2}\ln|t| + C_1}$

$v'(t) = C_2\sqrt{t}$

$v(t) = \frac{2}{3}Ct^{\frac{3}{2}} + K$

Seperable Wevra Problem

Find System of eq with X_1 / X_2

$X' = AX => \begin{pmatrix} 2t \\ 2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} t^2 \\ 2t \end{pmatrix}; \quad \begin{pmatrix} e^t \\ e^t \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e^t \\ e^t \end{pmatrix}$

$2t = at^2 + b2t$ → eq1

$2 = ct^2 + d2t$

$ae^t + be^t = e^t$ → eq2

$ce^t + de^t = e^t$

eq3. eq4.

$at^2 = 2t - b(2t)$

$a = \frac{2}{t} - \frac{2b}{t}$

$\left(\frac{2}{t} - \frac{2b}{t}\right)e^t + be^t = e^t$

Plug to eq 2 →

$\frac{2}{t}e^t - \frac{2b}{t}e^t + be^t = e^t$

$-\frac{2b}{t}e^t + be^t = e^t - \frac{2}{t}e^t$

$b\left(-\frac{2}{t}e^t + e^t\right) = \left(e^t - \frac{2}{t}e^t\right)$

$b = 1$

$X_1 = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$

$X_2 = \begin{pmatrix} e^t \\ e^t \end{pmatrix}$

if $t_0 \geq 0$, the Solution exist for $t \in [0, \infty]$

if $t_0 < 0$: the Solution exists for $t \in (-1, \infty)$.

23. $y'' + 2y' + y = 4e^{-t}; \quad y(0) = 2, \quad y'(0) = -1$

$s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + Y(s) = \frac{4}{s+1}$

$s^2Y(s) - 2s + 1 + 2sY(s) - 4 + Y(s) = \frac{4}{s+1}$

$Y(s)(s^2 + 2s + 1) = \frac{4}{s+1} + 2s + 3$

$Y(s) = \frac{4}{(s+1)(s+1)^2} + \frac{2s}{(s+1)^2} + \frac{3}{(s+1)^2}$

$= \frac{4}{(s+1)^3} + \frac{A}{(s+1)} + \frac{B}{(s+1)^2} = \frac{2s}{(s+1)^2}$

$A(s+1) + B = 2s$

$As + A + B = 2s$

$A + B = 0$

$2 + B = 0 \quad A = 2$

$= \frac{4}{(s+1)^3} + \frac{2}{(s+1)} + \frac{-2}{(s+1)^2} + \frac{3}{(s+1)^2}$

$Y(s) = \frac{4}{(s+1)^3} + \frac{2}{(s+1)} + \frac{1}{(s+1)^2}$

$Y(t) = 2e^{-t}t^2 + 2e^{-t} + e^{-t}t$