Definition: Piecewise continuous functions

A function f is called **piecewise continuous over the interval** $I \subset \mathbb{R}$ if f is defined over I and continuous at every point of I except for a finite set of points $t_1, \ldots, t_N \in I$, at which f has finite left and right limits.

Transform table

Example

$$f(t) := \begin{cases} 0, & \text{if } t < 0 \\ 1, & \text{if } t \geq 0. \end{cases}$$

Function f is discontinuous, and we can prove that there exists no differentiable function x such that $x'(t) = f(t)$ for all $t \in \mathbb{R}$. However, there are situations in which we may want to interpret and solve such an equation. Ideas?

Definition: Improper integral

Let f be a function defined on an unbounded interval $[a, \infty)$. If the integral $\int_a^T f(t)\mathrm{d}t$ is defined for all $T > a$ and admits a finite limit as $T \to \infty$, then we say that the **improper integral** of f over $[a, \infty)$ is **convergent** and write:

$$\int_a^\infty f(t)\mathrm{d}t := \lim_{T\to\infty} \int_a^T f(t)\mathrm{d}t.$$

If the limit is infinite or does not exist, then the improper integral $\int_a^\infty f(t)\mathrm{d}t$ **diverges**.

Remark

- We can have a convergent $\int_a^\infty f(t)\mathrm{d}t$ while $f(t)$ does not converge to 0.
- If $\int_a^\infty f(t)\mathrm{d}t$ converges, then so does the series $\sum a_n$, with $a_n := \int_n^{n+1} f(t)\mathrm{d}t$.
- The converse is not true: take $f(t) = \sin(2\pi t)$.
- If $f(t) \geq 0$ for all t larger than some M, then $T \mapsto \int_a^T f(t)\mathrm{d}t$ is an increasing function of T, and thus has a limit $L \in \mathbb{R} \cup \{\infty\}$ as $T \to \infty$. (We still say that

Theorem: Comparison

Let f, g be piecewise continuous functions defined over $[a, \infty)$ satisfying, for all t large enough, $0 \leq f(t) \leq g(t)$. Then:

$$\int_a^\infty g(t)\mathrm{d}t < \infty \quad \Rightarrow \quad \int_a^\infty f(t)\mathrm{d}t < \infty$$
$$\int_a^\infty f(t)\mathrm{d}t = \infty \quad \Rightarrow \quad \int_a^\infty g(t)\mathrm{d}t = \infty.$$

Definition: Absolute convergence

If $\int_a^\infty |f(t)|\mathrm{d}t < \infty$, then $\int_a^\infty f(t)\mathrm{d}t$ is convergent. We say that $\int_a^\infty f(t)\mathrm{d}t$ is **absolutely convergent**.

Remark

A convergent improper integral is not necessarily absolutely convergent: *e.g.*, $\int_a^\infty \frac{\sin t}{t}\mathrm{d}t$.

2 improper integral to remember

1.) $\int_0^\infty e^{-ct}\,dt \Rightarrow \begin{cases} \frac{1}{c} & \text{if } c > 0 \\ \infty, & \text{if } c \leq 0 \end{cases}$

$= \lim_{T\to\infty}\left[-\frac{1}{c}e^{-ct}\right]_0^T \Rightarrow \lim_{T\to\infty} \frac{1}{c}[1 - e^{-cT}] = \frac{1}{c}$ for $c > 0$

if $c \leq 0 \to +\infty$

2.) $\int_0^\infty t^s\,dt = \begin{cases} \frac{1}{s+1}, & \text{if } s < -1 \\ \infty, & \text{if } s \geq -1. \end{cases}$

$\lim_{T\to\infty}\left[\frac{t^{s+1}}{s+1}\Big|_0^T\right] = \frac{t^{s+1} - 0^{s+1}}{s+1}$

$\to \infty$ if $s+1 > 0$

$\to \frac{1}{s+1}$ if $s+1 < 0$

$L(1) = \frac{1}{s}$

$L(e^{at}) = \frac{1}{s-a}$

$L(t^n) = \frac{n!}{s^{n+1}}$

$L(\sin(at)) = \frac{a}{s^2+a^2}$

$L(\cos(at)) = \frac{s}{s^2+a^2}$

$L(f'(t)) = sF(s) - f(0)$

$L(f''(t)) = s^2F(s) - sF(0) - f'(0)$

$L(\cosh(at))(s) = \frac{s}{s^2-a^2}, \quad s > |a|$

$L(\sinh(at))(s) = \frac{a}{s^2-a^2}, \quad s > |a|$

- In all the applications, we assume that the solution y of the studied ODE satisfies the hypotheses that allow us to take its Laplace transform: y is piecewise continuous and of exponential order, and so are all of its derivatives involved in the ODE.
- When we find y, we need to either check that y solves the ODE...
- ... or check that y satisfies the conditions that justify the use of the Laplace transform.

Step function (Heaviside function)

$U_c(t) = U(t-c) = \begin{cases} 0 & 0 \leq t < c \\ 1 & t \geq c \end{cases}$ off to on

or $H(t)$ $H(t-c)$

Formula

$hU_c(t) = \begin{cases} 0 \\ h \end{cases}$

$1 - U_c(t) = \begin{cases} 1 \\ 0 \end{cases}$

$h(1 - U_c(t)) = \begin{cases} h \\ 0 \end{cases}$

on to off

If $c = 3$ (switch occur at $c = 3$)

then $\to$ $U_3(t) = U(t-3) = \begin{cases} 0 & 0 \leq t < 3 \\ 1 & t \geq 3 \end{cases}$

Ex. $f(t) = \begin{cases} -2 & 0 \leq t < 1 \\ 4 & 1 \leq t < 3 \\ 8 & 3 \leq t < 5 \\ 12 & t \geq 5 \end{cases}$ $\to t=1$ $U_1(t)$; $t=3$ $U_3(t)$; $t=5$ $U_5(t)$ break pt.

$f(t) = -2 + 6U_1(t) + 4U_3(t) + 4U_5(t)$

Can multiply by constant $\to$ $f(t)U_c(t) = \begin{cases} 0 \\ f(t) \end{cases}$

Exercise

Express function f as a linear combination of Heaviside functions, and compute its Laplace transform:

$f(t) = \begin{cases} 1, & t < 3, \\ -3, & 3 \leq t < 4, \\ 2, & 4 \leq t < 6, \\ 4, & 6 \leq t < 7. \end{cases}$

$f(t) = 1 - 4U_3(t) + 5U_4(t) + 2U_6(t)$

Definition: Laplace Transform

Let f be a function such that the integral below is defined for some $s \in \mathbb{R}$. The **Laplace transform** of $f(t)$ is function $F(s)$, defined as

$$\mathcal{L}(f)(s) = F(s) := \int_0^\infty e^{-st}f(t)\mathrm{d}t \qquad (1)$$

always assume $a - s < 0$

for all s for which the improper integral converges. We denote: $F = \mathcal{L}(f)$.

Note that $\mathcal{L}(f)$ only depends on the values of $f(t)$ for $t \geq 0$.

$$\mathcal{L}(u_0)(s) = \mathcal{L}(1)(s) = \frac{1}{s}, \qquad s > 0,$$
$$\mathcal{L}(u_c)(s) = \mathcal{L}(u_0(t-c))(s) = \frac{e^{-cs}}{s}, \qquad s > 0.$$

Convolution.

Theorem

Let f be a function defined over $\mathbb{R}$ such that:

- f is piecewise continuous over every interval $[0, T]$, $T > 0$,
- f is of exponential order: there exist constants $K > 0$, $M > 0$ and $a \in \mathbb{R}$ such that
$$\forall t \geq M, \quad |f(t)| \leq Ke^{at}.$$

Then the Laplace transform $F := \mathcal{L}(f)$ is defined for all $s > a$.

Definition

Let g and h be piece-wise continuous functions defined for $t \geq 0$. We define their convolution f for $t \geq 0$ as

$$f(t) := (g * h)(t) = \int_0^t g(\tau)h(t-\tau)\mathrm{d}\tau, \quad t \geq 0.$$

Example $\quad h(t-\tau) = t - \tau$

Let $g(t) = t^2$ and $h(t) = t$. We have:

$$g * h(t) = \int_0^t g(\tau)h(t-\tau)\mathrm{d}\tau = \int_0^t \tau^2(t-\tau)\mathrm{d}\tau$$
$$= t\int_0^t \tau^2\mathrm{d}\tau - \int_0^t \tau^3\mathrm{d}\tau = \frac{t^4}{3} - \frac{t^4}{4} = \frac{t^4}{12}.$$

Laplace transform by Integration

$L(tf(t)) = -\frac{d}{ds}L(f(t))\,ds$

$L(t^n f(t)) = \frac{d^n}{d^n s}L(f(t))\,ds$

not the same as $f(t)$!! (No negative)

$\cos at \Rightarrow \frac{e^{iat} + e^{-iat}}{2}$

In each of Problems 1 through 12:
(a) Find the solution of the given initial value problem.
(b) Draw a graph of the solution.

1. $y'' + 2y' + 2y = \delta(t - \pi)$; $\quad y(0) = 1, \quad y'(0) = 0$

1.) $y'' + 2y' + 2y = \delta(t - \pi)$; $y'' + 2y' + 2y = e^{-\pi s}$

$s^2Y(s) - sy(0) - y'(0) + 2sY(s) - 2y(0) + 2Y(s) = e^{-\pi s}$

$s^2Y(s) - s + 2sY(s) - 2 + 2Y(s) = e^{-\pi s}$

$Y(s)(s^2 + 2s + 2) = e^{-\pi s} + s + 2$

$Y(s) = \frac{2}{(s+1)^2+1} + \frac{s}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}e^{-\pi s}$

Manipulate to get $s+1$ at the top.

$Y(s) = \frac{2}{(s+1)^2+1} + \frac{(s+1)-1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}(e^{-\pi s})$

$Y(s) = \frac{2}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}\cdot(e^{-\pi s})$

$Y(s) = \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}(e^{-\pi s})$

$y(t) \Rightarrow e^{-t}\sin t + e^{-t}\cos(t) + (e^{-(t-\pi)}\sin(t - \pi))u_\pi(t)$

10.) $F(s) = \frac{2s-3}{s^2+2s+10}$ $= \frac{2s-3}{(s+1)^2+9} = \frac{2s+2-5}{(s+1)^2+9} = \frac{2(s+1)}{(s+1)^2+9} - \frac{5}{(s+1)^2+9}$

$2e^{-t}\cos(3t) - \frac{5}{3}e^{-t}\sin(3t)$

29. Suppose that f and f' are continuous for $t \geq 0$ and of exponential order as $t \to \infty$. Use integration by parts to show that if $F(s) = \mathcal{L}\{f(t)\}$, then $\lim_{s\to\infty} F(s) = 0$. The result is actually true under less restrictive conditions, such as those of Theorem 6.1.2.

→ Table of transform

Exercise

Find the Laplace transform of

$$f(t) = \begin{cases} \frac{\sin(t)}{t}, & t > 0, \\ 0, & t = 0. \end{cases}$$

$\int_0^\infty e^{-st}f(t)\,dt \Rightarrow \mathcal{L}(f)(s) = \frac{\pi}{2} - \arctan(s)$, $s > 0$

$F(s) := \mathcal{L}(f)(s) = \int_0^\infty e^{-st}\frac{\sin(t)}{t}\,dt$

$F'(s) = \frac{d}{ds}\left(\int_0^\infty e^{-st}\frac{\sin t}{t}\,dt\right) = \int_0^\infty \frac{d}{ds}(e^{-st})\frac{\sin(t)}{t}\,dt = \int_0^\infty \frac{-t}{t}e^{-st}\sin(t)\,dt$

$= -\mathcal{L}(\sin)(s) = \frac{-1}{s^2+1^2}$, $s > 0$

$F'(s) = -\frac{1}{s^2+1} \Rightarrow F(s) = -\int \frac{1}{1+s^2}\,ds = C - \arctan(s)$

$F(s) \xrightarrow[s\to\infty]{} 0$ because f is bounded. So: $C = \lim_{s\to\infty}\arctan(s) = \frac{\pi}{2}$.

6.2 Solution of initial Value problem

Inverse table transformation.

$\mathbf{f(t) = \mathcal{L}^{-1}(F(s))}$	$\mathbf{F(s) = \mathcal{L}(f(t))}$
$e^{at}t^n, \quad n \in \mathbb{N}$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
$e^{at}f(t)$	$F(s-a)$
$cf(ct)$	$F\left(\frac{s}{c}\right)$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$
f'	$sF(s) - f(0)$
f''	$s^2F(s) - sf(0) - f'(0)$
$f^{(n)}$	$s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{(k-1)}(0)$
$(-t)^nf$	$F^{(n)}(s)$

$L(u(t-c)) = \frac{e^{-cs}}{s}$

$L(f(t-c)u(t-c)) = e^{-cs}F(s)$

$L(g(t)u(t-c)) = e^{-cs}L(g(t+c))$

Step function table

Ex. $y'' + 4y' = (\sin t)u(t - 2\pi)$ $\quad y(0) = 1$; $y'(0) = 0$

let $g(t) = \sin t$, $c = 2\pi$

$g(t + 2\pi) = \sin(t + 2\pi)$ $\quad \sin(t+2\pi) = \sin(t)$

Laplace transform for step function

Transform $s^2Y(s) - sy(0) - y'(0) + 4(sY(s) - y(0)) = e^{-2\pi s}L(\sin(t))$

$= s^2Y(s) - s + 4sY(s) - 4 = e^{-2\pi s}\frac{1}{s^2+1}$

$Y(s) = \frac{1}{s} + \frac{1}{s(s+4)(s^2+1)}e^{-2\pi s}$

$= \frac{1}{s} + \left(\frac{A}{s} + \frac{B}{s+4} + \frac{Cs+D}{s^2+1}\right)e^{-2\pi s}$

Impulse Function

The Dirac delta δ_a admits a Laplace transform, and for $a > 0$, we have

$$\mathcal{L}(\delta_a)(s) = \int_0^\infty e^{-st}\delta_a(t)\mathrm{d}t = e^{-sa}.$$

The case $a = 0$ is more delicate, since the Laplace transform is defined on $\mathbb{R}_+$.

Example

We have

$$\mathcal{L}\left(\sum \alpha_i\delta_{a_i}\right)(s) = \sum \alpha_i e^{-a_i s}.$$

Solve the IVP

$$y'' + 3y' + 2y = \delta_1 - 2\delta_2, \qquad y(0) = 1;\ y'(0) = -1.$$

Let $Y = \mathcal{L}(y)$. Then:

$$\mathcal{L}(y') = sY(s) - y(0) = sY(s) - 1,$$
$$\mathcal{L}(y'') = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s + 1.$$

We get:

$$(s^2 + 3s + 2)\,Y(s) = e^{-s} - 2e^{-2s} + s + 2,$$

hence:

$$Y(s) = \frac{e^{-s}}{s^2+3s+2} - \frac{2e^{-2s}}{s^2+3s+2} + \frac{s+2}{s^2+3s+2}.$$

We have: $\frac{1}{s^2+3s+2} = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$, so:

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+3s+2}\right) = \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+2}\right) = e^{-t} - e^{-2t}.$$

Hence:

$$\mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2+3s+2}\right) = u_1(t)\left(e^{-(t-1)} - e^{-2(t-1)}\right),$$
$$\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s^2+3s+2}\right) = u_2(t)\left(e^{-(t-2)} - e^{-2(t-2)}\right),$$
$$\mathcal{L}^{-1}\left(\frac{s+2}{s^2+3s+2}\right) = \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) = e^{-t}.$$

26. $y'' + 4y = \begin{cases} t, & 0 \le t < 1, \\ 1, & 1 \le t < \infty; \end{cases}$ $y(0) = 0, \quad y'(0) = 0$

Convert

$L(s) = s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \int_0^1 e^{-st}(t)\,dt + \int_1^\infty e^{-st}\,dt$

$\frac{1}{s^2} - \frac{e^{-s}}{s} - \frac{e^{s}}{s^2}$; $\frac{e^{-s}}{s}$

Integrating factor for first ODE - s (b)

$Y'(s) + s^2Y(s) = S$; $P(s)$, $f(x)$

$I = e^{\int P(x)dx} = e^{\int s^2 ds} = e^{\frac{s^3}{3}}$

Multiply the whole integral by I =>

$e^{\frac{s^3}{3}}(Y'(s)) + e^{\frac{s^3}{3}} s^2(Y(s)) = Se^{\frac{s^3}{3}}$

$\frac{d}{ds}\left(Y(s)e^{\frac{s^3}{3}}\right) = Se^{\frac{s^3}{3}}$

$e^{\frac{s^3}{3}}Y(s) = \int re^{\frac{r^3}{3}}\,dr + C_1$

Hence transform => $Y(s) = e^{\frac{-s^3}{3}}\left(\int_0^s re^{\frac{r^3}{3}}\,dr + C_1\right)$

(b) Show that $Y(s) = c(1+s^2)^{-1/2}$, where c is an arbitrary constant.

From question a. $(s^2+1)Y'(s) + sY(s) = 0$

$(s^2+1)Y'(s) = -sY(s)$

$\frac{Y'(s)}{Y(s)} = -\frac{s}{s^2+1}$

$\ln|Y(s)| = -\frac{1}{2}\ln(s^2+1) + C_1$

$e^{\ln|Y(s)|} = e^{\ln(s^2+1)^{-\frac{1}{2}} + C_1}$

$Y(s) = C_1(1+s^2)^{-\frac{1}{2}}$

36. Consider Bessel's equation of order zero

$$ty'' + y' + ty = 0.$$

Recall from Section 5.7 that $t = 0$ is a regular singular point for this equation, and therefore solutions may become unbounded as $t \to 0$. However, let us try to determine whether there are any solutions that remain finite at $t = 0$ and have finite derivatives there. Assuming that there is such a solution $y = \phi(t)$, let $Y(s) = \mathcal{L}\{\phi(t)\}$.

(a) Show that $Y(s)$ satisfies

$$(1 + s^2)Y'(s) + sY(s) = 0.$$

(b) Show that $Y(s) = c(1 + s^2)^{-1/2}$, where c is an arbitrary constant.

(c) Writing $(1 + s^2)^{-1/2} = s^{-1}(1 + s^{-2})^{-1/2}$, expanding in a binomial series valid for $s > 1$, and assuming that it is permissible to take the inverse transform term by term, show that

$$y = c\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{2^{2n}(n!)^2} = cJ_0(t),$$

where J_0 is the Bessel function of the first kind of order zero. Note that $J_0(0) = 1$ and that J_0 has finite derivatives of all orders at $t = 0$. It was shown in Section 5.7 that the second solution of this equation becomes unbounded as $t \to 0$.

(a) As $Y(s) = L\{\phi(t)\}$ and $ty'' + y' + ty = 0$ are linear; then we can apply Laplace transform

$L(y(s)) = L(ty'') + L(y') + L(ty) = 0$

$L(tf(t)) = -\frac{d}{ds}L(f(t))ds = -\frac{d}{ds}L(y'') + L(y') - \frac{d}{ds}L(y)$

$\Rightarrow -\frac{d}{ds}\left[s^2Y(s) - sy(0) - y'(0)\right] + sY(s) - y(0) - \frac{d}{ds}(Y(s)) = 0$

$= -2sY(s) - s^2Y'(s) + y(0) + sY(s) - y(0) - Y'(s) = 0$

$-sY(s) - (s^2+1)Y'(s) =$

Hence Prove → $= (s^2+1)Y'(s) + sY(s)$

the sides of the Taylor series expansion for $\sin t$.

$\mathcal{L}\{\sin t\} = \mathcal{L}\left\{\sum_{n=0}^\infty \frac{(-1)^n t^{2n+1}}{(2n+1)!}\right\}$

$= \sum_{n=0}^\infty \mathcal{L}\left\{\frac{(-1)^n t^{2n+1}}{(2n+1)!}\right\}$

$= \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!}\mathcal{L}\{t^{2n+1}\}$

$= \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!}\frac{(2n+1)!}{s^{(2n+1)+1}}$

$= \sum_{n=0}^\infty \frac{(-1)^n}{s^{2n+2}}$

$= \frac{1}{s^2}\sum_{n=0}^\infty \frac{(-1)^n}{s^{2n}}$

$= \frac{1}{s^2}\sum_{n=0}^\infty \left(-\frac{1}{s^2}\right)^n$

$= \frac{1}{s^2}\frac{1}{1-\left(-\frac{1}{s^2}\right)}$

$= \frac{1}{s^2+1}$

(a) Show that $F'(s) = \mathcal{L}\{-tf(t)\}$.

(b) Show that $F^{(n)}(s) = \mathcal{L}\{(-t)^n f(t)\}$; hence differentiating the Laplace transform corresponds to multiplying the original function by $-t$.

Differentiate both sides with respect to s.

$F'(s) = \frac{d}{ds}\int_0^\infty e^{-st}f(t)\,dt$

$= \int_0^\infty \frac{\partial}{\partial s}e^{-st}f(t)\,dt$

$= \int_0^\infty (-te^{-st})f(t)\,dt$

$= \int_0^\infty e^{-st}[-tf(t)]\,dt$

$= \mathcal{L}\{-tf(t)\}$

Part (b)

Differentiate both sides n times with respect to s.

$F^{(n)}(s) = \frac{d^n}{ds^n}\int_0^\infty e^{-st}f(t)\,dt$

$= \int_0^\infty \frac{\partial^n}{\partial s^n}e^{-st}f(t)\,dt$

$= \int_0^\infty [(-t)^n e^{-st}]f(t)\,dt$

$= \int_0^\infty e^{-st}[(-t)^n f(t)]\,dt$

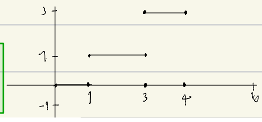
$= \mathcal{L}\{(-t)^n f(t)\}$

In each of Problems 1 through 6, sketch the graph of the given function on the interval $t \ge 0$.

1. $g(t) = u_1(t) + 2u_3(t) - 6u_4(t)$
2. $g(t) = (t-3)u_2(t) - (t-2)u_3(t)$
3. $g(t) = f(t-\pi)u_\pi(t)$, where $f(t) = t^2$
4. $g(t) = f(t-3)u_3(t)$, where $f(t) = \sin t$
5. $g(t) = f(t-1)u_2(t)$, where $f(t) = 2t$
6. $g(t) = (t-1)u_1(t) - 2(t-2)u_2(t) + (t-3)u_3(t)$

1.) $g(t) = u_1(t) + 2u_3(t) - 6u_4(t)$

$g(t) = H(t-1) + 2H(t-3) - 6H(t-4)$



Laplace for $t - u_1(t)(t-1)$

$F(s) = \int_0^\infty e^{-st}[t - (t-1)H(t-1)]\,dt$

$= \int_0^\infty te^{-st}\,dt - \int_0^\infty (t-1)e^{-st}H(t-1)\,dt$

$= \int_0^\infty te^{-st}\,dt - \int_1^\infty (t-1)e^{-st}\,dt$

$= \int_0^\infty te^{-st}\,dt - \int_1^\infty te^{-st}\,dt + \int_1^\infty e^{-st}\,dt$

$= \int_0^\infty\left(-\frac{\partial}{\partial s}e^{-st}\right)dt - \int_1^\infty\left(-\frac{\partial}{\partial s}e^{-st}\right)dt + \int_1^\infty e^{-st}\,dt$

38. Suppose that

$$g(t) = \int_0^t f(\tau)\,d\tau.$$

If $G(s)$ and $F(s)$ are the Laplace transforms of $g(t)$ and $f(t)$, respectively, show that

$$G(s) = F(s)/s.$$

$G(s) = L(g(t)) = \int_0^\infty e^{-st}\left[\int_0^t f(\tau)\,d\tau\right]dt$

Laplace transform → $L(g(t)) = e^{-st}$

$\Rightarrow \int_0^\infty\int_\tau^\infty e^{-st}f(\tau)\,dt\,d\tau$

$\Rightarrow \int_0^\infty\left(\frac{1}{s}e^{-st}\Big|_\tau^\infty\right)f(\tau)\,d\tau$

$\frac{1}{s}\int_0^\infty e^{-s\tau}f(\tau)\,d\tau$

$\frac{1}{s}L(f(t)) \Rightarrow \frac{F(s)}{s}$

Transform of $t^2\sin 6t$.

!!!

$L(f(ct)) = \frac{1}{c}F\left(\frac{s}{c}\right)$

$F(s) = \int_0^\infty e^{-st}t^2\sin bt\,dt$

$= \int_0^\infty\left(\frac{\partial^2}{\partial s^2}e^{-st}\right)\sin bt\,dt$

$= \frac{d^2}{ds^2}\int_0^\infty e^{-st}\sin bt\,dt$

$= \frac{d^2}{ds^2}\mathcal{L}\{\sin bt\}$

$= \frac{d^2}{ds^2}\left(\frac{b}{s^2+b^2}\right)$

$= \frac{d}{ds}\left[-\frac{2bs}{(s^2+b^2)^2}\right]$

$= -\frac{(2b)(s^2+b^2)^2 - 2bs[2(s^2+b^2)\cdot 2s]}{(s^2+b^2)^4}$

$= -\frac{2b(s^4+2s^2b^2+b^4) - 8bs^2(s^2+b^2)}{(s^2+b^2)^4}$

$= -\frac{2b^5 - 4b^3s^2 - 6bs^4}{(s^2+b^2)^4}$

$= \frac{2b(3s^2-b^2)(s^2+b^2)}{(s^2+b^2)^4}$

$= \frac{2b(3s^2-b^2)}{(s^2+b^2)^3}$

Part (a)

Suppose we use $f(ct)$ rather than $f(t)$.

$$\mathcal{L}\{f(ct)\} = \int_0^\infty e^{-st}f(ct)\,dt.$$

Make the substitution $x = ct$. Then $dx = c\,dt$.

$\mathcal{L}\{f(ct)\} = \int_0^\infty e^{-sx/c}f(x)\left(\frac{dx}{c}\right)$

$= \frac{1}{c}\int_0^\infty e^{-(s/c)x}f(x)\,dx$

$= \frac{1}{c}F\left(\frac{s}{c}\right)$

14. Find an expression involving $u_c(t)$ for a function f that ramps up from zero at $t = t_0$ to the value h at $t = t_0 + k$.

15. Find an expression involving $u_c(t)$ for a function g that ramps up from zero at $t = t_0$ to the value h at $t = t_0 + k$ and then ramps back down to zero at $t = t_0 + 2k$.

$g(t) = \frac{h}{k}(t-t_0)H(t-t_0) - \frac{h}{k}(t-t_0)H(t-t_0-k)$
$+ \frac{h}{k}(t_0+2k-t)H(t-t_0-k) - \frac{h}{k}(t_0+2k-t)H(t-t_0-2k)$

$$f(t) = \frac{h}{k}(t-t_0)H(t-t_0) - \frac{h}{k}(t-t_0)H(t-t_0-k) + hH(t-t_0-k)$$

For the first term, the Heaviside function $H(t-t_0)$, defined to be 1 if $t > t_0$ and 0 if $t < t_0$, make the function start increasing at $t = t_0$. The factor of $(h/k)(t-t_0)$ makes it so that it reaches a height of h at $t = t_0 + k$. The second term, which activates when $t = t_0 + k$, cancels the first term. The third term makes $f(t)$ have a value of h from $t = t_0 + k$ and on. $f(t)$ can also be written as

Spring Mass System

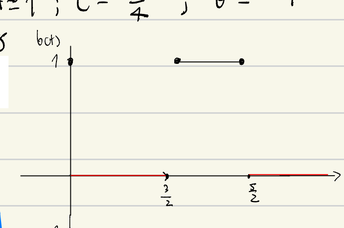
$u'' + \frac{1}{4}u' + u = kg(t), \qquad u(0) = 0, \quad u'(0) = 0,$

$= \left\{k - ke^{-(t-\frac{3}{2})/8}\cos\left[\frac{\sqrt{63}}{8}\left(t-\frac{3}{2}\right)\right] - \frac{k}{\sqrt{63}}e^{-(t-\frac{3}{2})/8}\sin\left[\frac{\sqrt{63}}{8}\left(t-\frac{3}{2}\right)\right]\right\}H\left(t-\frac{3}{2}\right)$

$-\left\{k - ke^{-(t-\frac{5}{2})/8}\cos\left[\frac{\sqrt{63}}{8}\left(t-\frac{5}{2}\right)\right] - \frac{k}{\sqrt{63}}e^{-(t-\frac{5}{2})/8}\sin\left[\frac{\sqrt{63}}{8}\left(t-\frac{5}{2}\right)\right]\right\}H\left(t-\frac{5}{2}\right)$

$A = 1$; $C = -\frac{1}{4}$; $B = -1$

$t = 25.68$



graph of $g(t)$

$F(s) = \int_0^\infty e^{-st}t^n e^{at}\,dt$

$= \int_0^\infty\left[(-1)^n\frac{\partial^n}{\partial s^n}e^{-st}\right]e^{at}\,dt$

$= (-1)^n\frac{d^n}{ds^n}\int_0^\infty e^{-st}e^{at}\,dt$

$= (-1)^n\frac{d^n}{ds^n}\int_0^\infty e^{(a-s)t}\,dt$

$= (-1)^n\frac{d^n}{ds^n}\left[\frac{1}{a-s}e^{(a-s)t}\Big|_0^\infty\right]$

$= (-1)^n\frac{d^n}{ds^n}\left(\frac{1}{s-a}\right)$

$= (-1)^n\frac{d^n}{ds^n}[(s-a)^{-1}]$

$= (-1)^n[(-1)^n n!(s-a)^{-1-n}]$

$= (-1)^{2n}\frac{n!}{(s-a)^{n+1}}$

$= \frac{n!}{(s-a)^{n+1}}$

$t^n e^{at}$

(b) Show that if k is a positive constant, then

$$\mathcal{L}^{-1}\{F(ks)\} = \frac{1}{k}f\left(\frac{t}{k}\right).$$

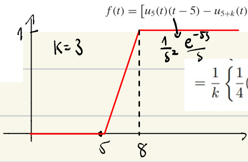
(c) Show that if a and b are constants with $a > 0$, then

$$\mathcal{L}^{-1}\{F(as+b)\} = \frac{1}{a}e^{-bt/a}f\left(\frac{t}{a}\right).$$

Modify the problem in Example 2 of this section by replacing the given forcing function $g(t)$ by

$f(t) = [u_5(t)(t-5) - u_{5+k}(t)(t-5-k)]/k.$

(a.) $k = 3$



$\frac{1}{s^2}\frac{e^{-5s}}{s}$; $\frac{1}{s^2}\frac{e^{(5-k)s}}{s}$

Solve initial Value Problem

$= \frac{1}{k}\left\{\frac{1}{4}(t-5) - \frac{1}{8}\sin[2(t-5)]\right\}u_5(t) - \frac{1}{k}\left\{\frac{1}{4}(t-5-k) - \frac{1}{8}\sin[2(t-5-k)]\right\}u_{5+k}(t)$

Impulse Ex.

4. $y'' - y = -20\delta(t-3); \qquad y(0) = 1, \quad y'(0) = 0$

$s^2Y(s) - sy(0) - y'(0) - Y(s) = -20e^{-3s}$

$s^2Y(s) - s - Y(s) = -20e^{-3s}$

$= Y(s)(s^2-1) = -20e^{-3s} + S$

$Y(s) = -20\left(\frac{1}{s^2-1}\right)e^{-3s} + \frac{s}{s^2-1}$

$y(t) = -20\sinh(t-3)u_3(t) + \cosh(t)$

Consider the initial value problem

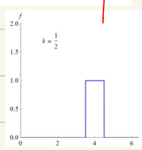
$$y'' + \frac{1}{3}y' + 4y = f_k(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

where

$$f_k(t) = \begin{cases} 1/2k, & 4-k \le t < 4+k \\ 0, & 0 \le t < 4-k \quad \text{and} \quad t \ge 4+k \end{cases}$$

graph

$f_k(t) \Rightarrow \frac{1}{2k}u_{(4-k)}(t) - \frac{1}{2k}u_{(4+k)}(t)$



$[s^2Y(s) - sy(0) - y'(0)] + Y(s) = \int_0^\infty e^{-st}u_0(t)\,dt + 2\sum_{k=1}^n(-1)^k\int_0^\infty e^{-st}u_{k\pi}(t)\,dt$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 0$.

$[s^2Y(s)] + Y(s) = \int_0^\infty e^{-st}\,dt + 2\sum_{k=1}^n(-1)^k\int_{k\pi}^\infty e^{-st}\,dt$

$(s^2+1)Y(s) = \frac{1}{s} + 2\sum_{k=1}^n(-1)^k\left(\frac{e^{-k\pi s}}{s}\right)$

Solve for $Y(s)$.

$Y(s) = \frac{1}{s(s^2+1)} + 2\sum_{k=1}^n(-1)^k\left[\frac{1}{s(s^2+1)}\right]e^{-k\pi s}$

Now write it in terms of known transforms by using partial fraction decomposition.

$Y(s) = \left(\frac{1}{s} - \frac{s}{s^2+1}\right) + 2\sum_{k=1}^n(-1)^k\left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-k\pi s}$

$y(t) = \mathcal{L}^{-1}\{Y(s)\}$

$= \mathcal{L}^{-1}\left\{\left(\frac{1}{s} - \frac{s}{s^2+1}\right) + 2\sum_{k=1}^n(-1)^k\left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-k\pi s}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+1}\right\} + 2\sum_{k=1}^n(-1)^k\mathcal{L}^{-1}\left\{\left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-k\pi s}\right\}$

$= 1 - \cos t + 2\sum_{k=1}^n(-1)^k[1 - \cos(t-k\pi)]H(t-k\pi)$

$= 1 - \cos t + 2\sum_{k=1}^n(-1)^k[1 - \cos(t-k\pi)]u_{k\pi}(t)$

Integrating factor

$y = \frac{1}{I(x)}\left[\int I(x)Q(x)\,dx + C\right]$ → from $y' + P(x)y = Q(x)$

Convolution Ex.

$$f(t) = \int_0^t (t-\tau)^2\cos 2\tau\,d\tau$$

Solution

Recognize that $f(t)$ is a convolution integral of the two functions, t and $\cos 2t$. The Laplace transform of $f(t)$ can be found by using the convolution theorem, which states that

$$\mathcal{L}\left\{\int_0^t g(t-\tau)h(\tau)\,d\tau\right\} = G(s)H(s),$$

where G and H are the Laplace transforms of g and h, respectively. Therefore,

$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{\int_0^t (t-\tau)^2\cos 2\tau\,d\tau\right\}$

$= \mathcal{L}\{t^2\}\mathcal{L}\{\cos 2t\}$

$= \left(\frac{2!}{s^{2+1}}\right)\left(\frac{s}{s^2+4}\right)$

$= \frac{2s}{s^3(s^2+4)}$

$= \frac{2}{s^2(s^2+4)}$

Seperable

Ex. Find $\frac{dy}{dx} = \frac{\sec^2 x}{\tan^2 y}$

$\tan^2 y\,dy = \sec^2 x\,dx$

$\int \tan^2 y\,dy = \int \sec^2 x\,dx$

$\int \sec^2 x - 1 \quad : \quad \tan x$

$\tan(y) - y = \tan x + C$

Note: * $y' = e^{x-y} \Rightarrow y' = (e^x)(e^{-y})$

$= \frac{1}{e^{-y}}dy = e^x\,dx$

Exact

$(2x-1)\,dx + (3y+7)\,dy = 0$

$\frac{\partial M}{\partial y} = 0, \quad \frac{\partial N}{\partial x} = 0$ exact

Hence $\int M\,dx = \frac{2x^2}{2} - x + g(y)$

$\int N\,dy = \frac{3y^2}{2} + 7y + f(x)$

Hence $C = x^2 - x + \frac{3}{2}y^2 + 7y$ Ans

Second Order ODE

$y'' + 9y = 0 \qquad M = 3i \quad = 0 \pm 3i$

$M^2 + 9 = 0$

$M^2 = -9$

$M = \sqrt{-9}$

$y = C_1\cos(3x) + C_2\sin(3x)$

$\mathcal{L}\{t^2\sin t\} = \int_0^\infty e^{-st}t^2\sin(t)\,dt$

we want the case when $s = 1$, so plug $s = 1$ into answer from part (a)

$\frac{6 \cdot 1^2 - 2}{(1^2+1)^3}$

$= \frac{4}{8}$

$\frac{1}{2}$

Linearisation.

$x'(t) = x(2-x-y)$

$y'(t) = -x + 3y - 2xy$

at $x = 0 \rightarrow y = 0$

as $2 - x - y = 0$

$y = 2 - x$

sub to eq 2: $-x + 3(2-x) - 2x(2-x) = 0$

$-x + 6 - 3x - 4x + 2x^2 = 0$

$2x^2 - 8x + 6 = 0$

$x^2 - 4x + 3 = 0$

$(x-3)(x-1) = 0$

$x = 3, \; x = 1$

at $x = 3 \rightarrow y = -1$, at $x = 1 \rightarrow y = 1$

Crit Point. => $(0,0)$ $(3,-1)$ $(1,1)$

linearisation. →

$\begin{vmatrix} \frac{\partial f}{\partial x}x'_1 & \frac{\partial f}{\partial y}x'_1 \\ \frac{\partial g}{\partial x}y_1 & \frac{\partial g}{\partial y}y'_1 \end{vmatrix} = \begin{vmatrix} 2-2x-y & -x \\ -1-2y & 3-2x \end{vmatrix}$ $-1-(3)$

$X(1,1) = \begin{vmatrix} -1 & -1 \\ -3 & 1 \end{vmatrix}$ $\lambda^2 - 4 = 0$

$\lambda^2 = 4 \rightarrow \lambda = \pm 2$

as $\lambda_1 < 0$ and $\lambda_2 > 0$ it is Unstable

Unit Step function

If $f(t) = \begin{cases} 4t & \text{if } t < 2, \\ t^2 + 4 & \text{if } 2 \le t, \end{cases}$

$f(t) \Rightarrow 4t + (t^2 - 4t + 4)U_2(t)$

or $4t + (U(t-2))(t-2)^2$

* $\frac{v''}{v'} = \frac{1}{2t}$

$\frac{d}{dt}(\ln(v'(t))) = \frac{1}{2t}$

$\ln(v'(t)) = \int \frac{1}{2t}\,dt$

$\ln|v'(t)| = \frac{1}{2}\ln(t) + C$

$|v'(t)| = e^{\frac{1}{2}\ln|t| + C_1}$

$v'(t) = C_2\sqrt{t}$

$v(t) = \frac{2}{3}Ct^{\frac{3}{2}} + K$

Separable wern Problem

Find System of eq with X_1 / X_2

$X' = AX \Rightarrow \begin{pmatrix} 2t \\ 2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} t^2 \\ 2t \end{pmatrix}$; $\begin{pmatrix} e^t \\ e^t \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e^t \\ e^t \end{pmatrix}$

$2t = at^2 + b2t$ → eq1

$2 = ct^2 + d2t$

$ae^t + be^t = e^t$ → eq2

$ce^t + de^t = e^t$ eq4

$at^2 = 2t - b(2t)$ eq3

$a = \frac{2}{t} - \frac{2b}{t}$

$\left(\frac{2}{t} - \frac{2b}{t}\right)e^t + be^t = e^t$

Plug to eq2 →

$\frac{2}{t}e^t - \frac{2b}{t}e^t + be^t = e^t$

$-\frac{2b}{t}e^t + be^t = e^t - \frac{2}{t}e^t$

$b\left(-\frac{2}{t}e^t + e^t\right) = \left(e^t - \frac{2}{t}e^t\right)$

$b = 1$

$X_1 = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$

$X_2 = \begin{pmatrix} e^t \\ e^t \end{pmatrix}$

If $t_0 \ge 0$, the Solution exist for $t \in [0, \infty]$

if $t_0 < 0$: the Solution exists for $t \in (-1, \infty)$.

23. $y'' + 2y' + y = 4e^{-t}; \qquad y(0) = 2, \quad y'(0) = -1$

$s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + Y(s) = \frac{4}{s+1}$

$s^2Y(s) - 2s + 1 + 2sY(s) - 4 + Y(s) = \frac{4}{s+1}$

$Y(s)(s^2 + 2s + 1) = \frac{4}{s+1} + 2s + 3$

$Y(s) = \frac{4}{(s+1)(s+1)^2} + \frac{2s}{(s+1)^2} + \frac{3}{(s+1)^2}$

$= \frac{4}{(s+1)^3}$

$\frac{A}{(s+1)} + \frac{B}{(s+1)^2} = \frac{2s}{(s+1)^2}$

$A(s+1) + B = 2s$

$As + A + B = 2s$

$A + B = 0$

$2 + B = 0 \qquad A = 2$

$= \frac{4}{(s+1)^3} + \frac{2}{(s+1)} + \frac{-2}{(s+1)^2} + \frac{3}{(s+1)^2}$

$Y(s) = \frac{4}{(s+1)^3} + \frac{2}{(s+1)} + \frac{1}{(s+1)^2}$

$Y(t) = 2e^{-t}t^2 + 2e^{-t} + e^{-t}t$